MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

VOL. XXVII.

FROM JANUARY TO JUNE, 1877.

C.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1877.

LONDON:
PRINTED BY C. F. HODGSON & SON,
GOUGH SQUARE, FLEET STREET.

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CORRIGENDA.

VOL. XXVII.

p. 24, line 13, *for* $\frac{d(u^m)}{d\theta^2}$ *read* $\frac{d(u^m)}{d\theta}$.

p. 56, line 1, *for* OMAKYA *read* BYOMAKESA.

p. 80, line 6, *for* "inverse" *read* "reciprocal."

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5129. (By Sir JAMES COCKLE, F.R.S.)—Find three cases in which

$$r - a^2y^{2b}t = 0 \quad\text{..............................(1)}$$

is soluble without changing either variable.

I. *Solution by the* PROPOSER.

1. The symbolical product

$$\left(\frac{d}{dx} + \alpha\frac{d}{dy} + \gamma\right)\left(\frac{d}{dx} + \beta\frac{d}{dy} + \epsilon\right)$$

is equivalent to

$$\frac{d^2}{dx^2} + (\alpha+\beta)\frac{d^2}{dx\,dy} + \alpha\beta\frac{d^2}{dy^2} + (\gamma+\epsilon)\frac{d}{dx} + \left(\beta\gamma + \alpha\epsilon + \frac{d\beta}{dx} + \alpha\frac{d\beta}{dy}\right)\frac{d}{dy}$$
$$+ \gamma\epsilon + \frac{d\epsilon}{dx} + \alpha\frac{d\epsilon}{dy},$$

which, applied to the operand z, gives rise to a partial biordinal, say,

$$r + 2Ss + Tt + Pp + Qq + Zz = 0,$$

wherein $\quad \alpha+\beta = 2S, \quad \alpha\beta = T, \quad \gamma+\epsilon = P$ ....(2, 3, 4);

$$\beta\gamma + \alpha\epsilon + \frac{d\beta}{dx} + \alpha\frac{d\beta}{dy} = Q, \quad \gamma\epsilon + \frac{d\epsilon}{dx} + \alpha\frac{d\epsilon}{dy} = Z \quad\text{......... (5, 6).}$$

Hence, taking (1) as the biordinal, we may put $\alpha = ay^b$, $\beta = -ay^b$, and $\gamma = -\epsilon$; so that (5) and (6) become

$$2ay^b\epsilon - a^2by^{2b-1} = 0, \quad \frac{d\epsilon}{dx} + ay^b\frac{d\epsilon}{dy} = \epsilon^2 \quad\text{.............. (7, 8).}$$

Now when $b=0$, then $\epsilon=0$, $\gamma=0$, $\alpha=a$, and $\beta=-a$; so that, since either sign may be given to a, we have

$$\left(\frac{d}{dx} \pm a\frac{d}{dy}\right)\left(\frac{d}{dx} \mp a\frac{d}{dy}\right)z = 0$$

as the symbolical decomposition of (1); and the complete solution is therefore

$$z = \phi(y + ax) + \psi(y - ax).$$

2. Suppose that ϵ is a function of y only. Then, by (8), $-\dfrac{1}{\epsilon} = -\dfrac{y^{1-b}}{a(1-b)}$, and (7) becomes $2a^2(b-1) = a^2b$; whence $b = 2$, and the symbolical decomposition of $r - a^2y^4t = 0$ is

$$\left(\frac{d}{dx} \pm ay^2\frac{d}{dy} \mp ay\right)\left(\frac{d}{dx} \mp ay^2\frac{d}{dy} \pm ay\right)z = 0;$$

whence the two first integrals $p \pm ay^2q \mp ayz = 0$, and thence the complete solution. Thus far Monge's method would apply. We now come to the third case.

3. Write (1) in the form

$$0 = \left(\frac{d}{dx} + ay^b\frac{d}{dy}\right)\left(\frac{d}{dx} - ay^b\frac{d}{dy}\right)z + a^2by^{2b-1}\frac{dz}{dy};$$

then a symbolical decomposition can be effected if we can find a function f such that it will satisfy

$$a^2by^{2b-1}\frac{dz}{dy} = \left(\frac{d}{dx} + ay^b\frac{d}{dy}\right)(x, y) \quad \text{.....................} \quad (9)$$

and a certain further condition. Let $f = f\left(x + \dfrac{y^{1-b}}{a(1-b)}\right)$; then $\dfrac{df}{dx} = f'$, and $\dfrac{df}{dy} = \dfrac{f'}{ay^b}$; therefore $\dfrac{df}{dx} = ay^b\dfrac{df}{dy}$. Consequently (9) becomes

$$a^2by^{2b-1}\frac{dz}{dy} = 2ay^b\frac{df}{dy}, \quad \text{or} \quad \frac{dz}{dy} = \frac{2}{ab}y^{1-b}\frac{df}{dy};$$

whence, by integration with respect to y, we obtain

$$z = \frac{2}{ab}\left\{y^{1-b}f - \int\frac{(1-b)f}{y^b}dy\right\} \quad \text{.....................} \quad (10).$$

But when (9) is satisfied (1) is reducible to

$$\left(\frac{d}{dx} - ay^b\frac{d}{dy}\right)z = -f \quad \text{.......................} \quad (11);$$

whence we deduce $x + \dfrac{y^{1-b}}{a(1-b)} = c_1$, and $\dfrac{z}{f} - \dfrac{y^{1-b}}{a(1-b)} = c_2$.

Hence we have

$$z = \frac{y^{1-b}}{a(1-b)}f + F \quad \text{..............................} \quad (12),$$

where f and F are arbitrary functions with the same argument, viz., $x + \dfrac{y^{1-b}}{a(1-b)}$. Now, in order that (10) and (12) may coincide, we must have

$$\frac{2}{b} = \frac{1}{1-b}, \quad \text{or} \quad b = \tfrac{2}{3};$$

and also $F = -\dfrac{2}{ab}\displaystyle\int\frac{(1-b)f}{y^b}dy$, or $f = -ay^b\dfrac{dF}{dy} = -F'\left(x + \dfrac{y^{1-b}}{a(1-b)}\right)$.

Hence a particular solution of $r - a^2y^{\frac{4}{3}}t$ is

$$z = F\left(x + \frac{3y^{\frac{1}{3}}}{a}\right) - 3\frac{y^{\frac{1}{3}}}{a}F'\left(x + \frac{3y^{\frac{1}{3}}}{a}\right) \quad \text{..............} \quad (13),$$

a result which I shall verify thus:

$$r = F'' - \frac{3y^{\frac{1}{3}}}{a} F''', \quad q = -\frac{3}{a^2 y^{\frac{4}{3}}} F'', \quad t = \frac{1}{a^2 y^{\frac{4}{3}}} F'' - \frac{3}{a^2 y} F''';$$

therefore $r - a^2 y^{\frac{4}{3}} t = 0$. From the fact that we can give either sign to a, and the consequent reversibility of factors, the complete solution of this last equation is

$$z = \phi\left(x + \frac{3y^{\frac{1}{3}}}{a}\right) + \psi\left(x - \frac{3y^{\frac{1}{3}}}{a}\right) - \frac{3y^{\frac{1}{3}}}{a}\left\{\phi'\left(x + \frac{3y^{\frac{1}{3}}}{a}\right) - \psi'\left(x - \frac{3y^{\frac{1}{3}}}{a}\right)\right\}.$$

II. *Solution by* Professor Lloyd Tanner, M.A.

A slight modification of a method of mine (*Messenger of Mathematics*, Vol. V., p. 53) will suffice to solve (1), without changing either of the variables, in any one of the infinite number of cases in which it is known to admit of a finite solution. In the paper cited, it is shown as follows:—

(α). That the argument of one of the arbitrary functions in the solution

$$= \frac{1}{(b-1)\, y^{b-1}} + ax \ (= u \text{ say});$$

the other being formed by changing the sign of a:

(β). That the solution is of the form

$$z = u_0 \phi(u) + u_1 \phi'(u) + \ldots\ldots + u_n \phi^{(n)}(u)$$
$$+ \text{another series of arbitrary functions}:$$

(γ). That, for this equation (1), $u_0, u_1, \ldots u_n$ can be determined as functions of y only:

(δ). That when $0 \leqq b < 1$ the number of derivatives of the arbitrary functions is $\frac{1}{2} \cdot \frac{b}{1-b}$; when $1 < b \leqq 2$ this number is $\frac{2-b}{2(b-1)}$; a finite solution being possible only when one of these expressions is a positive integer.

To illustrate the method, I take the case of $b = \frac{4}{3}$, not given by Sir James Cockle. The equation

$$r - a^2 y^{\frac{8}{3}} t = 0 \quad \ldots\ldots (2)$$

admits of a solution of the form

$$z = u_0 \phi(3y^{-\frac{1}{3}} + ax) + u_1 \phi'(3y^{-\frac{1}{3}} + ax) \quad \ldots\ldots (3).$$

Now, u_0, u_1 being functions of y only, we have

$$r = a^2 u_0 \phi'' + a^2 u_1 \phi''',$$

$$t = \frac{d^2 u_0}{dy^2}\phi + \left\{\frac{4}{3} y^{-\frac{7}{3}} u_0 - 2y^{-\frac{4}{3}} \frac{du_0}{dy} + \frac{d^2 u_1}{dy}\right\}\phi'$$
$$+ \left\{y^{-\frac{8}{3}} u_0 + \frac{4}{3} y^{-\frac{7}{3}} u_1 - 2y^{-\frac{4}{3}} \frac{du_1}{dy}\right\}\phi'' + y^{-\frac{8}{3}} u_1 \phi'''.$$

On substituting in (2), the function ϕ''' disappears. The coefficient of ϕ'' gives a relation for u_1; viz., $\frac{du_1}{dy} - \frac{2}{3} y u_1 = 0$,

the solution of which is $u_1 = k y^{\frac{2}{3}}$.

Equating to zero the coefficients of ϕ', ϕ, and reducing, we find u_0 must

satisfy the equations $\dfrac{du_0}{dy} - \dfrac{2}{3}\cdot\dfrac{u_0}{y} + \dfrac{k}{9} = 0, \quad \dfrac{d^2u_0}{dy^2} = 0;$

which give $$u_0 = -\tfrac{1}{3}ky.$$

Hence (2) becomes, on putting $k = -3$,

$$z = y\phi(3y^{-\frac{1}{3}} + ax) - 3y^{\frac{4}{3}}\phi'(3y^{-\frac{1}{3}} + ax),$$

and the complete solution, found by observing that the sign of a is indifferent, is

$$z = y\left\{\phi(3y^{-\frac{1}{3}} + ax) + \psi(3y^{-\frac{1}{3}} - ax)\right\}$$
$$- 3y^{\frac{4}{3}}\left\{\phi'(3y^{-\frac{1}{3}} + ax) + \psi'(3y^{-\frac{1}{3}} - ax)\right\}.$$

Of course the solution becomes much simpler if, as was done in the paper I have quoted, the variables are changed; but the above example will indicate that this change of variables is a convenient rather than an essential part of the method.

5130. (By Professor Cayley.)—Show that the envelop of a variable circle having its centre on a given conic and cutting at right angles a given circle is a bicircular quartic; which, when the given conic and circle have double contact, becomes a pair of circles; and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems—

(1). Given two conics, to find a polygon of n sides inscribed in the one and circumscribed about the other.

(2). Given two circles, to find a closed series of n circles each touching the two circles and the two adjacent circles of the series.

Solution by Professor Townsend, M.A., F.R.S.

The two celebrated porisms referred to in the above may be readily connected, without reference to the general, by consideration of the particular case only, as follows:—

It is well known [see Townsend's *Modern Geometry*, Art. 193] that a variable circle touching two fixed circles intersects at right angles a third fixed circle coaxal with the two; and it is evident that its variable centre describes a bifocal conic whose foci are the centres of the two. Hence, as the several centres of a closed system, when such is possible, of any number of derived circles, touching each an original pair and the two adjacent of each other, determine evidently a polygon at once inscribed to the bifocal conic and circumscribed to the orthogonal circle corresponding to the pair; therefore, &c.

5054. (By Professor Wolstenholme, M.A.)—If a circle be drawn on a chord of the parabola $y^2 = 4ax$ which passes through the point $(b, 0)$,

prove that the envelop of the circle will be the circular cubic

$$(x^2+y^2)(x+a)-x^2(a+2b)+bx(b-4a)=0.$$

The foci of this are the points on the axis whose distances from the origin are $b-4a$, b, $-b\pm 2a^{\frac{1}{2}}b^{\frac{1}{2}}$; and if r_1, r_2, r_3 be the distances of any point on the envelop from these points, show that

$$2r_3a^{\frac{1}{2}}=r_1b^{\frac{1}{2}}+r_2(2a^{\frac{1}{2}}+b^{\frac{1}{2}}).$$

When $b=4a$, two foci coincide at the origin, and we have a nodal curve. When $b=a$, two pairs of foci coincide at a, $-3a$; hence two nodes and the cubic breaks up into the straight line $r_1=r_2$ and the circle $3r_1=r_2$. Two foci coincide in no other case.

Solution by Prof. Nash, M.A.; Prof. Evans; *and others.*

The equation of the circle described upon the chord $y=m(x-b)$ is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2),$$

where x_1, x_2 are the roots of $m^2x^2-2x(m^2b+2a)+m^2b^2=0$, and y_1, y_2 the roots of $my^2-4ay-4abm=0$; therefore the equation of the circle is

$$x^2-2x(b+2am^{-2})+b^2+y^2-4am^{-1}y-4ab=0 \;\;\ldots\ldots\ldots\ldots (1).$$

Since this is a quadratic in the indeterminate m^{-1}, the envelop is

$$16a^2y^2+16ax(x^2+y^2-2bx+b^2-4ab)=0,$$

or $$(x^2+y^2)(x+a)-x^2(a+2b)+bx(b-4a)=0 \;\;\ldots\ldots\ldots\ldots\ldots (2),$$

a circular cubic whose axis is the axis of x. The four real foci are found from the condition that the line $y=(-1)^{\frac{1}{2}}(x-x')$ should touch the cubic. The abscissæ of the four foci are the roots of the equation

$$x^4+4ax^3-2b^2x^2+4abx(b-4a)+b^2(b-4a)^2=0 \;\;\ldots\ldots\ldots\ldots (3).$$

The roots of this equation are b, $b-4a$, $-b\pm 2a^{\frac{1}{2}}b^{\frac{1}{2}}$. If $b=4a$, two foci coincide with the origin, and the others are at distances $4a$ and $-8a$ from it.

If $b=a$, the equation becomes $(x^2+2ax-3a^2)^2=0$, and two foci coincide at a and also two at $-3a$. The cubic meets the axis in the points 0, $b\pm 2a^{\frac{1}{2}}b^{\frac{1}{2}}$, and the distances of the last two points from the foci are

$$2a^{\frac{1}{2}}(b^{\frac{1}{2}}+2a^{\frac{1}{2}}),\;\; 2a^{\frac{1}{2}}b^{\frac{1}{2}},\;\; 2b^{\frac{1}{2}}(2a^{\frac{1}{2}}+b^{\frac{1}{2}}),$$

$$2a^{\frac{1}{2}}(b^{\frac{1}{2}}-2a^{\frac{1}{2}}),\;\; 2a^{\frac{1}{2}}b^{\frac{1}{2}},\;\; 2b;$$

therefore, if $l\rho_1+m\rho_2+n\rho_3=0$ be the equation referred to the foci, we have

$$l\,a^{\frac{1}{2}}(b^{\frac{1}{2}}+2a^{\frac{1}{2}})+m\,a^{\frac{1}{2}}b^{\frac{1}{2}}+n\,b^{\frac{1}{2}}(b^{\frac{1}{2}}+2a^{\frac{1}{2}})=0;$$

$$l\,a^{\frac{1}{2}}(b^{\frac{1}{2}}-2a^{\frac{1}{2}})+m\,a^{\frac{1}{2}}b^{\frac{1}{2}}+nb=0;$$

therefore $$\frac{l}{b^{\frac{1}{2}}}=\frac{m}{b^{\frac{1}{2}}+2a^{\frac{1}{2}}}=\frac{n}{-2a^{\frac{1}{2}}};$$

therefore the equation of the cubic is

$$b^{\frac{1}{2}}\rho_1+(2a^{\frac{1}{2}}+b^{\frac{1}{2}})\rho_2=2a^{\frac{1}{2}}\rho_3.$$

5147. (By E. B. Elliott, M.A.)—Prove that the cones which run from any point of a quadric surface to all plane sections of that surface have a

common pair of generators real, coincident, or imaginary; and cut any plane parallel to the tangent plane at their common vertex in a system of similar and similarly situated conics.

Solution by Professor LLOYD TANNER, M.A.

Take the given point as origin and the tangent plane to the quadric at the origin for plane of yz, so that equation of quadric becomes

$$u_2 + x = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where u_2 is homogeneous of 2nd degree in xyz. The equation of any surface passing through a plane section of (1) is

$$u_2 + x + \lambda(ax + by + cz - 1) = 0.$$

If this be a cone having its vertex at origin, we must have $\lambda = x$: thus the equations of the cones are of the form

$$u_2 + x\,(ax + by + cz) = 0.$$

If in this we put $x = \text{const.}$, we get the equations of conics; and the coefficients of the highest powers of y, z being the same in all, we have a set of similar and similarly placed conics.

These conics have two points in common, viz. at an infinite distance on their common asymptotes. The lines joining these two points to the origin are the common generators of the cones. In other words, these common generators are the asymptotes of the indicatrix to (1) at the origin.

[This is CHASLES' well known extension of the old familiar property of Stereographic projection from any point of a sphere on any parallel to the tangent plane at the point. See Note 28, in his celebrated *Aperçu Historique*, published originally at Brussels in 1837, and reprinted without alterations at Paris in 1875.]

5117. (By CHRISTINE LADD.)—Prove that tangents from the two external or the two imaginary intersections of diagonals of the quadrilateral formed by tangents to a conic, from any two vertices of a self-conjugate triangle, intersect again on sides of the triangle.

Solution by the PROPOSER.

The equation to the conic referred to the self-conjugate triangle is $l^2\alpha^2 + m^2\beta^2 = n^2\gamma^2$. The tangents from vertices are

$$(A)\ l\alpha \pm n\gamma = 0, \quad (B)\ m\beta \pm n\gamma = 0, \quad (C)\ l\alpha \pm \sqrt{(-1)}\, m\beta = 0.$$

The diagonals of the quadrilateral (AB) are γ, $l\alpha \pm m\beta$; of (BC) are β, $l\alpha \pm \sqrt{(-1)}\, n\gamma$; of (CA) are α, $m\beta \pm \sqrt{(-1)}\, n\gamma$. The tangents through the external intersections of (AB) and the imaginary intersections of (BC) and (CA) are (AB) $\sqrt{2}\, n\gamma \pm l\alpha \pm m\beta = 0$, (BC) $2l\alpha \pm \sqrt{2}\, n\gamma \pm \sqrt{(-2)}\, m\beta = 0$, (CA) $2\,m\beta \pm \sqrt{2}\, n\gamma \pm \sqrt{(-2)}\, l\alpha = 0$. It appears from these equations that the lines (AB) intersect again on α and β, the lines (BC) on β and γ, and the lines (CA) on γ and α.

5165. (By R. A. ROBERTS, M.A.)—Prove that the envelop of the director circles of a series of conics passing through four fixed points is a bicircular quartic of which the vertices of the fixed triangle that is self-conjugate with regard to the conics are foci.

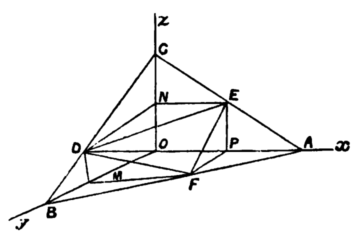
Solution by J. C. MALET, M.A.

Since the locus of the centre of the variable director circle is a conic passing through the vertices of the fixed self-conjugate triangle, and since it also cuts orthogonally the circle circumscribing the same triangle, its envelop is a bicircular quartic of which the double foci are the foci of the fixed conic, and four single foci are the intersections of the fixed circle and conic. [See CASEY'S *Bicircular Quartics.*]

5071. (By ST. JOHN STEPHEN.)—If three straight lines are drawn at random in a plane; show that the probability of their forming an acute-angled triangle is $\frac{1}{4}$.

Solution by the PROPOSER.

1. We shall first of all have recourse to a geometrical illustration for the solution of this problem. Let x, y, z be the angles formed by the lines; then we have the relation $x+y+z=\pi$. Draw the plane $\frac{x}{\pi}+\frac{y}{\pi}+\frac{z}{\pi}=1$, and let it cut the coordinate axes in A, B, and C. Bisect OA, OB, OC by planes EPF, DMF, DNE parallel to the planes of yz, zx. xy respectively, and join DE, EF, FD. Then, since AC = CB = BA, we have



$$\Delta AEF = \Delta ECD = \Delta DBF = \Delta DEF.$$

Now within each of the triangles AEF, ECD, DBF one of the coordinates is greater than $\frac{1}{2}\pi$, and within the triangle DEF each of them is less than $\frac{1}{2}\pi$. Therefore the chance the lines should form an acute-angled triangle $=\frac{\Delta DEF}{\Delta ABC}=\frac{\Delta DEF}{4\Delta DEF}=\frac{1}{4}$. Similarly, the chance that they formed an obtuse-angled triangle would be $\frac{3}{4}$.

2. These results might also have been arrived at by simply considering the different cases. The cases in which any two of the lines are at right angles, or are parallel, may be left out of consideration, as the chances for these two are indefinitely small. We however know that any two may form an obtuse angle, and therefore we have three obtuse-angled triangles likely to be formed. And there can only be one acute-angled triangle. Thus there are four cases possible, three obtuse and one acute, and the chances of these will be the same as we obtained before.

4758. (By the EDITOR.)—Find the value of each of the following expressions:—

$$\int_0^{\frac{1}{2}\pi} \tan\theta \log \operatorname{cosec}\theta \, d\theta; \quad (\cos m\theta)^{\operatorname{cosec}^2 n\theta}, \text{ when } \theta = 0; \quad \frac{d^6 \cos^{\frac{1}{2}}\theta}{d\theta^6} \ldots (1, 2, 3).$$

Solution by S. WATSON; Prof. EVANS, M.A.; *and others.*

1. Put $\cos\theta = x$; then $\tan\theta\, d\theta = -\dfrac{dx}{x}$; hence the required integral

$$= -\tfrac{1}{2}\int_0^1 \frac{dx}{x}\log_e(1-x^2) = \tfrac{1}{2}\int_0^1 (x + \tfrac{1}{2}x^3 + \tfrac{1}{3}x^5 + \tfrac{1}{4}x^7 + \&c.)\, dx$$

$$\tfrac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right) = \frac{\pi^2}{24}.$$

2. Put $u = (\cos m\theta)^{\operatorname{cosec}^2 n\theta}$; then $\log u = \dfrac{\log(\cos m\theta)}{\sin^2 n\theta} = \dfrac{0}{0}$ when $\theta = 0$;

but, after differentiating twice in the usual manner, we have

$$-\frac{m^2 \sec^2 m\theta}{2n^2 \cos^2 n\theta}, \text{ which, when } \theta = 0, \text{ becomes } -\frac{m^2}{2n^2};$$

therefore the required value is $u = \log^{-1}\left(-\dfrac{m^2}{2n^2}\right)$.

3. Put $u^2 = \cos\theta$, then we have

$$\frac{du}{d\theta} = -\frac{\sin\theta}{2u}, \text{ and } \frac{d(u^m)}{d\theta^2} = mu^{m-1}\frac{du}{d\theta} = -\tfrac{1}{2}mu^{m-2}\sin\theta;$$

therefore $$\frac{d^2(u^m)}{d\theta^2} = -\frac{m}{2}\left\{(m-2)u^{m-3}\sin\theta\frac{du}{d\theta} + u^{m-2}\cos\theta\right\}$$

$$= -\frac{m}{4}\left\{mu^m - (m-2)u^{m-4}\right\}.$$

By the application of this formula, we easily obtain

$$\frac{d^2u}{d\theta^2} = -\tfrac{1}{4}(u + u^{-3}), \quad \frac{d^4u}{d\theta^4} = -\tfrac{1}{16}(u + 10u^{-3} - 15u^{-7}),$$

$$\frac{d^6u}{d\theta^6} = -\tfrac{1}{64}(u + 91u^{-3} - 885u^{-7} + 945u^{-10});$$

and so on as far as we please.

5069. (By R. A. ROBERTS, B.A.)—Prove that the locus of the triple foci of a series of Cartesian ovals passing through five fixed points is an equilateral hyperbola.

Solution by the Rev. F. D. Thomson, M.A.

The equation to a Cartesian whose triple focus is at (x, y) is of the form
$$[X^2+Y^2-2Xx-2Yy+c]^2=aX+bY+d;$$
or, writing P for $X^2+Y^2-2Xx-2Yy$, $P^2+2cP+c^2=aX+bY+d$.

Let the origin and the points 1, 2, 3, 4 be the 5 given points; then $c^2=d$, and the equation becomes $P^2+2cP=aX+bY$.

Hence, substituting the coordinates of 1, 2. 3, 4, and eliminating c, a, b, the locus of (x, y) is
$$\begin{vmatrix} P_1^2, & P_2^2, & P_3^2, & P_4^2 \\ P_1, & P_2, & P_3, & P_4 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \end{vmatrix}=0.$$

But
$$\begin{vmatrix} P_2, & P_3, & P_4 \\ x_2, & x_3, & x_4 \\ y_2, & y_3, & y_4 \end{vmatrix}=\begin{vmatrix} r_2^2, & r_3^2, & r_4^2 \\ x_2, & x_3, & x_4 \\ y_2, & y_3, & y_4 \end{vmatrix} \text{ identically;}$$

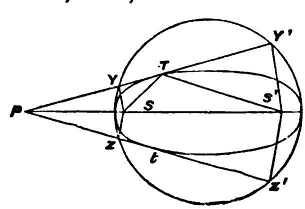
thus the locus becomes the conic
$$\begin{vmatrix} P_1^2, & P_2^2, & P_3^2, & P_4^2 \\ r_1^2, & r_2^2, & r_3^2, & r_4^2 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \end{vmatrix}=0.$$

$$\text{Coefft. of } x^2=4\begin{vmatrix} x_1^2, & x_2^2, & \dots \\ r_1^2, & r_2^2, & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}=4\begin{vmatrix} x_1^2, & x_2^2, & \dots \\ y_1^2, & y_2^2, & \dots \\ x_1, & x_2, & \dots \\ \dots & \dots & \dots \end{vmatrix}=-\text{coefft. of } y^2;$$

therefore conic is rectilinear hyperbola.

4765. (By S. A Renshaw.)—If the tangents PT, Pt to an ellipse be produced to meet the auxiliary circle in Y, Y′ and Z, Z′; and CD, CE be the semi-diameters respectively parallel to them, prove that
$$YT\,.\,TY'-Zt\,.\,tZ'=CD^2-CE'^2.$$

Solution by E. Rutter; C. Leudesdorf, M.A.; *and others.*



1. $TY\,.\,TY'$
$=SY\cot STY\,.\,S'Y'\cot S'TY'$
$=BC^2\cot^2 STY$
$=BC^2\operatorname{cosec}^2 STY-BC^2$
$=CD^2-BC^2$.

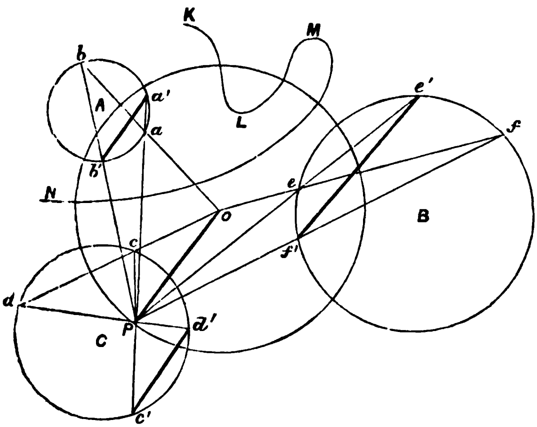
2. $tZ\,.\,tZ'=CE^2-BC^2$;

$\therefore TY\,.\,TY'-tZ\,.\,tZ'=CD^2-CE^2$.

4965. (By S. A. RENSHAW.)—A, B, C are three circles drawn in the same plane. Show that, in the same plane, a point and another circle may be found, such that, if through the point any three secants be drawn cutting the circles A, B, C respectively in the points a, b; c, d; e, f; then, if any point P be taken on the circle that it is stated can be found, and Pa, Pb; Pc, Pd; Pe, Pf be joined, cutting the circles A, B, C again in the points a', b'; c', d'; e', f', the lines $a'b$, $c'd'$, $e'f'$ will always be parallel to one another.

Solution by the PROPOSER.

1. The point to be found is O, the radical centre of the three given



circles, and the circle to be found is that which cuts the three given circles orthogonally, and has O for its centre. For, take one of the circles, *e.g.* A; then, since $b\text{O} \cdot \text{O}a = \text{OP}^2$, $ba : \text{OP} = \text{OP} : \text{O}a$; therefore the triangles bOP, POa are similar, and the angle PbO = aPO—that is, $b'a'a = a$PO; therefore $a'b'$ is parallel to OP; in the same way it may be shown that $c'd'$ and $e'f'$ are each of them parallel to OP. Hence $a'b'$, $c'd'$, $e'f'$ are all parallel to one another.

2. If the secants Oab, Ocd, Oef revolve about O, whilst the point P remains either fixed, or revolves in the circle O. Still $a'b'$, $c'd'$, $e'f'$ remain always parallel to one another.

3. Hence also, if KLMN be a curve of any degree whatever, the number of points on the curve that possess the property in question may be discovered, for they are evidently those in which the circle O cuts the curve; and the same is true when the curve KLMN becomes a straight line; in which case, if the intersections of the line with the circle O are real, there are two points on the line enjoying the property stated.

[Mr. RENSHAW remarks that the above theorem furnishes a species of parallel motion, and suggests that probably a similar theorem holds true for three spheres and a surface, &c.]

4983. (By Artemas Martin).—A speaks the truth b times out of a; B, d times out of c; and C, n times out of m. C says that B told him that A said a certain event happened; find the probability that the event happened.

Solution by the Proposer.

A statement coming from A through B and C, if A tells B and B tells C, is true if all speak truth, or if any two lie and the other speaks truth. [See my Solution of Quest. 4835.] The chance that all tell the truth is

$$\frac{b}{a} \times \frac{d}{c} \times \frac{n}{m} = \frac{bdn}{acm}, = p_1;$$

the chance that A speaks the truth and B and C lie is

$$\frac{b}{a}\left(1-\frac{d}{c}\right)\left(1-\frac{n}{m}\right), = p_2;$$

the chance that B speaks the truth and A and C lie is

$$\frac{d}{c}\left(1-\frac{b}{a}\right)\left(1-\frac{n}{m}\right), = p_3;$$

and the chance that C tells the truth and A and B lie is

$$\frac{n}{m}\left(1-\frac{b}{a}\right)\left(1-\frac{d}{c}\right), = p_4.$$

Therefore the chance in favour of the truth of a story coming from A through B and C, when A tells it to B and B tells it to C, is

$$p_1+p_2+p_3+p_4, \text{ which put } = \text{P}.$$

The statement is not true if all lie, or if any one of them lies and the other two tell the truth.

The chance that all lie is

$$\left(1-\frac{b}{a}\right)\left(1-\frac{d}{c}\right)\left(1-\frac{n}{m}\right), = q_1;$$

the chance that A and B speak truly and C falsely is

$$\frac{bd}{ac}\left(1-\frac{n}{m}\right), = q_2;$$

the chance that B and C speak truly and A falsely is

$$\frac{dn}{cm}\left(1-\frac{b}{a}\right), = q_3;$$

and the chance that A and C speak truly and B falsely is

$$\frac{bn}{am}\left(1-\frac{d}{c}\right), = q_4.$$

Therefore the chance against the truth of a story coming from A through B and C, when A tells it to B and B tells it to C, is

$$q_1+q_2+q_3+q_4, \text{ which put } = \text{Q}.$$

The sum of P and Q is unity, as it should be.

But we are not *sure* that A told B and B told C that the event occurred

—in fact the chance that A *did* tell B is $\frac{d}{c}$, and the chance that B told C is $\frac{n}{m}$; therefore the chance in favour of the event when C says that B said that A said it occurred, is

$$\frac{d}{c}\times\frac{n}{m}\times \mathrm{P}=\frac{dn}{cm}\left[\frac{bdn}{acm}+\frac{b}{a}\left(1-\frac{d}{c}\right)\left(1-\frac{n}{m}\right)\right.$$
$$\left.+\frac{d}{c}\left(1-\frac{b}{a}\right)\left(1-\frac{n}{m}\right)+\frac{n}{m}\left(1-\frac{b}{a}\right)\left(1-\frac{d}{c}\right)\right].$$

The event did not occur if A *did not* tell B, and B *did not* tell C, the chance of which is $1-\frac{dn}{cm}$; therefore the chance against the event is

$$\frac{d}{c}\times\frac{n}{m}\,\mathrm{Q}+\left(1-\frac{dn}{cm}\right)=\frac{dn}{cm}\left[\left(1-\frac{b}{a}\right)\left(1-\frac{d}{c}\right)\left(1-\frac{n}{m}\right)+\frac{bd}{ac}\left(1-\frac{n}{m}\right)\right.$$
$$\left.+\frac{dn}{cm}\left(1-\frac{b}{a}\right)+\frac{bn}{am}\left(1-\frac{d}{c}\right)\right]+\left(1-\frac{dn}{cm}\right).$$

The sum of these two chances is unity, as it should be.

See Note appended to my Solution of Quest. 4719.

4966. (By S. TEBAY, B.A.)—A circular disc can revolve freely in a vertical plane about its axis, while a heavy particle descends in a curvilinear groove between two points in its plane; find the brachystochrone.

Solution by the PROPOSER.

Let the higher point be the origin, m the mass of the particle, m' the mass of the disc, and θ the angle through which it has turned in the time t. Then, by the principle of *vis viva*, we have

$$m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}+k^2m\left(\frac{d\theta}{dt}\right)^2=2gmy\,;$$

or, putting $\quad p=\frac{dy}{d\theta},\quad p'=\frac{dx}{d\theta},\quad n^{-1}=k^2\frac{m'}{m},$

$$t=\int\left\{\frac{p^2+p'^2+n^{-1}}{2gy}\right\}^{\frac{1}{2}}dx,\quad\text{and}\quad \mathrm{V}=\left\{\frac{p^2+p'^2+n^{-1}}{y}\right\}^{\frac{1}{2}}.$$

Since p, p' are independent, we have

$$\frac{d\,(\mathrm{V})}{d\theta}=\mathrm{N}p+\mathrm{P}\frac{dp}{d\theta}+\mathrm{P}'\frac{dp'}{d\theta},\quad \mathrm{N}-\frac{d\,(\mathrm{P})}{d\theta}=0,\quad \frac{d\,(\mathrm{P}')}{d\theta}=0\,;$$

therefore $$\frac{d\,(\mathrm{V})}{d\theta}=\frac{d}{d\theta}(\mathrm{P}p)+\mathrm{P}'\frac{dp'}{d\theta},$$

and $$\mathrm{V}=\mathrm{P}p+\int\mathrm{P}'\frac{dp'}{d\theta}\,d\theta=\mathrm{P}p+\mathrm{P}'p'+\frac{1}{c},$$

integrating by parts.

But $\quad P = \dfrac{p}{\{y(p^2+p'^2+n^{-1})\}^{\frac{1}{2}}}, \quad P' = \dfrac{p'}{\{y(p^2+p'^2+n^{-1})\}^{\frac{1}{2}}};$

therefore $\quad p^2+p'^2+n^{-1} = \dfrac{n^2c^2}{y}, \quad P' = \dfrac{p'}{nc}, \quad \Theta = c'x,$

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2c^2c'^2}{y} - n^{-1}c'^2 - 1,$$

$$(n^{-1}c'^2+1)^{\frac{1}{2}}x = h \operatorname{vers}^{-1}\frac{y}{h} - \mathrm{V}(2hy-y^2),$$

putting $\dfrac{n^2c^2c'^2}{n^{-1}c'^2+1} = 2h$. The curve is therefore a cycloid. Let a be the distance of the higher point from the centre, α the angle it makes with the horizon, b the distance of the other point, and β the angle it makes with the horizon. Then

$$x = a\cos\alpha - b\cos(\beta+c'x), \quad y = a\sin\alpha + b\sin(\beta+c'x).$$

These equations, with the equation to the curve, furnish a relation between c, c'. Another relation is obtained from the condition that the cycloid cuts the circle described by the lower point at right angles, namely,

$$\left(\frac{n^2c^2c'^2}{y} - n^{-1}c'^2 - 1\right)^{\frac{1}{2}}\cot(\beta+c'x)+1 = 0.$$

5073. (By R. W. Genese, M.A.)—When the quadrilateral ABCD of Question 3523 is a maximum, prove that the orthocentre of BCD is the centre of curvature at C.

Solution by R. Tucker, M.A.; R. E. Riley, M.A.; *and others.*

Referring to the Solution of Question 3523 (*Reprint*, Vol. XVII., p. 27), we have Area ABCD $= \frac{1}{2}\sqrt{3}\,.\,ab\,[1+\cos(\frac{1}{3}\pi+\frac{4}{3}\delta)]$, which is a maximum when $\delta = -\frac{1}{4}\pi$. The coordinates of the orthocentre of BCD are

$$-\frac{a^2-b^2}{2a}\cos\delta, \quad \frac{a^2-b^2}{2b}\sin\delta,$$

and the coordinates of the centre of curvature at C are

$$\frac{a^2-b^2}{2a}\cos^3\left(\frac{2\pi}{3}-\frac{\delta}{3}\right), \quad -\frac{a^2-b^2}{2b}\sin^3\left(\frac{2\pi}{3}-\frac{\delta}{3}\right).$$

These are readily seen to be equal, for the above value of δ; therefore, &c.

4428. (By Professor Evans, M.A.)—Show that the product of five consecutive numbers of the natural series cannot be the square of a commensurable number.

Solution by the PROPOSER.

Let x represent the least of five consecutive integers. Every number that will divide any two of the five factors x, $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$ of the product $P_x = x(x+1)(x+2)(x+3)(x+4)$, will also divide their difference. As every divisor common to two of these five factors is therefore a term of the series 1, 2, 3, 4, the only prime factors common to any two of the five factors of P_x are 2 and 3.

If the condition $P_x = y^2$ were true, any one of the five factors of P_x not divisible by 2 or 3 would evidently be a square number. Moreover, no one of these five factors of P_x would be divisible by both 2 and 3; for if one of the extreme factors, x for example, were divisible by both 2 and 3, we should have $(x+2)$ and $(x+4)$ both divisible by 2, but neither divisible by 3; and therefore $x+2 = 2a^2$ and $x+4 = 2b^2$, whence b^2-a^2 would equal unity, which is impossible. The same demonstration applies to $(x+4)$ the other extreme factor; and if one of the three intermediate factors, as $(x+1)$, were divisible by both 2 and 3, x and $(x+2)$, which comprehend $(x+1)$ and differ from it by unity, would be divisible by neither 2 nor 3, and therefore x and $(x+2)$ would be squares, which is impossible, since they differ by only two units.

From the foregoing it is manifest that, if the condition $P_x = y^2$ existed, no one of the five factors of P_x would be divisible by 6, and that $x = 6m+1$, and $x+4 = 6m+5$, being divisible by neither 2 nor 3, would both be squares, which is manifestly impossible, since the difference of the squares of two integers cannot differ by less than five units.

4937. (By S. TEBAY, B.A.)—A straight rod slips down in a corner, the lower extremity moving along a straight line in the horizontal plane; determine the motion.

Solution by the PROPOSER.

Let $2a$ be the length of the rod, θ its inclination to the vertical, ϕ the inclination of its projection to the axis of x, xyz any point in the rod, and r its distance from the centre. Then

$$x = (a-r)\sin\theta\cos\phi,\quad y = (a-r)\sin\theta\sin\phi,\quad z = (a+r)\cos\theta.$$

By the principle of *vis viva*, we have

$$\int dr\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \text{const.} - 2g\int z\,dr,$$

or

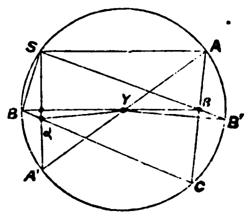
$$\left\{\frac{d}{dt}(\sin\theta\cos\phi)\right\}^2 + \left\{\frac{d}{dt}(\sin\theta\sin\phi)\right\}^2 + \left\{\frac{d}{dt}(\cos\theta)\right\}^2 - \frac{3g}{2a}(\cos\theta' - \cos\theta);$$

θ' being the initial value of θ.

Let $x\cos\alpha + y\sin\alpha = p$ be the straight line; or, putting $x = 2a\sin\theta\cos\phi$, $y = 2a\sin\theta\sin\phi$, $2a\sin\theta\cos(\phi-\mu) = p$. These, with the initial equation $2a\sin\theta'\cos(\phi'-\alpha) = p$, determine the motion.

4372. (By Professor KELLAND, M.A.)—ABC is a triangle inscribed in a circle; from any point O on the circumference perpendiculars are drawn to OA, OB, OC, meeting BC, CA, AB, in D, E, F: prove that D, E, F lie on a straight line which passes through the centre of the circle.

Solution by R. W. GENESE, M.A.



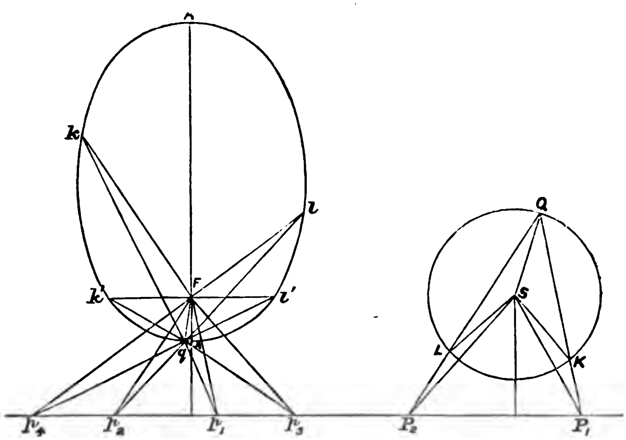
The theorem of which this is the reciprocal is easily proved as follows:—Let AA′, BB′ be two diameters of a circle S, C any two points on the circumference. Applying Pascal's theorem to the hexagon A′SB′BCA, we see that the intersections (A′S, BC), (SB′, CA), (B′B, AA′) are in one straight line. But SA′ is perpendicular to SA, SB′ to SB; and AA′, BB′ intersect in the centre of the circle, &c.

[See *Reprint*, Vol. XXVI., pp. 65, 92.]

4610. (By S. A. RENSHAW.)—Find conic theorems which are the analogues of Propositions 20, 21, 22, 32 of the Third Book of Euclid.

Solution by the PROPOSER.

1. Let AA′ be a conic of which F is one of the foci, and let S′ be the centre of a generating circle. Take any point Q in the circumference of

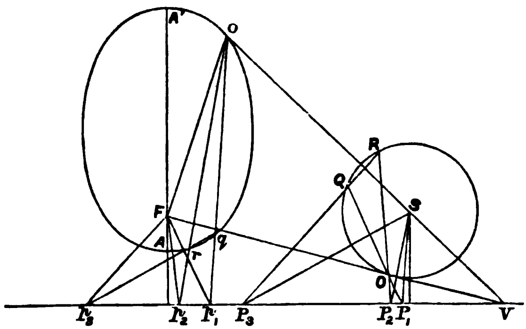


the circle, and draw QKP_1, QLP_2 cutting the circle again in K and L, and the directrix in P_1 and P_2, and join SP_1, SP_2. Then, drawing Fp_1, Fp_2 parallel to QP_1, QP_2, and p_1qk, p_2ql parallel to SP_1 and SP_2, these

lines pass through the same point q on the conic by the principle of the generating circle [see Art. 33 of Renshaw's *Conics*], and intersect the conic in two other points k and l; and, completing the figure, Fl is parallel to SL, Fk to SK, and Fq to SQ. Hence it is evident that the angle kFl = $2p_1$Fp_2. Therefore the conic analogue to Euc. III. 20 is this:—If k, l be two points on a conic, and q any third point, and if kq, lq be joined and produced to meet the directrix in p_1, p_2, and kF, lF, p_1F, p_2F be joined, the angle kFl = $2p_1$Fp_2.

When the points k and l are in the same straight line with F, that is, when k'Fl' is a focal chord, and $k'q$, $l'q$ meet the directrix in p_3, p_4 respectively, then the angle p_3Fp_4 is a right angle; and hence focal chords are analogous to diameters in the circle.

2. Let AA′ be a conic, and F one of its foci; also S be the centre of a generating circle. Draw the tangent FOV to the circle, and the tangent



VSo to the conic [see Art. 34 of Renshaw's *Conics*]; through O draw any line cutting the circle again in Q. Take any other point R in the circumference of the circle, and join RO, RQ, and produce them to meet the directrix in P_2, P_3, and let QO meet the same in P_1. Join SP_1, SP_2, SP_3, and draw Fp_1, Fp_2, Fp_3 parallel to QO, RO, QR, and meeting the directrix in p_1, p_2, p_3; and, lastly, draw through p_1, p_2, p_3 parallels to SP_1, SP_2, SP_3. Then, by the principle of the generating circle, p_1q, p_2r pass through o, and the parallel through p_3 to SP_3 passes through r and q. If, then, QO remaining fixed, the point R revolves in the circle and the point r in the conic, it is evident that the angle p_2Fp_3 will remain constant, being always equal to QRO or to p_1FV. Therefore the analogue to Euclid III. 21 will be as follows:—If oq be a chord of a conic, and r any third point thereon, and if qr, or be produced to meet the directrix in p_3, p_2, the angle p_2Fp_3 remains constant.

3. The analogue to Euclid III. 32 is as follows:—If oV be a tangent to a conic meeting the directrix in V, and FV be joined, and if through o any chord oq be drawn and produced to meet the directrix in p_1; then, if r be any other point in the conic, and or, qr be joined and produced to meet the directrix in p_2, p_3, and Fp_1, Fp_2, Fp_3 be joined, the angle p_2Fp_3 is always equal to the angle p_1FV.

If the circle be looked upon as a conic having its directrix situated at infinity, then it is evident that the properties of the circle (Euclid III. 21, 32) may also be enunciated as above; and the same is true of the theorem about the inscribed quadrilateral (Euclid III. 22).

JUST INTONATION.

By Colonel A. R. Clarke, C.B., F.R.S.

It is well known to all musicians,—though not, perhaps, sufficiently impressed on all students of music,—that the scale as played on an instrument with fixed tones, such as the pianoforte or organ, is not the true diatonic scale. With the exception of the octaves, the intervals are systematically mistuned, so that—in the system of Equal Temperament—all the scales are made equally good, or rather equally bad. The ear, from constant habit, grows accustomed to these false intervals, and does not ordinarily notice their defects.

Of late there has, however, been a tendency to be dissatisfied with this state of things; and various instruments of more or less complexity have accordingly been proposed or invented for the purpose of removing or diminishing the evil. One disadvantage common to them all is the large number of digitals required. From this, indeed, there is no escape; though it does not follow that the difficulties of playing are proportionally increased.

Amongst these instruments, the Voice-Harmonium invented by Mr. Colin Brown, Euing Lecturer on Music in the Andersonian University, Glasgow, is conspicuous for its simplicity, its elegance, and its uncompromising representation of true intervals. It is not only easy to understand, but presents great facilities to the performer. The keys are played in parallel lines, in the natural order of relationship, from that of C♭ which is nearest the performer, to that of C♯ which is the furthest off; the key of C holding a central position. The theory of the instrument is as follows:—

The vibration numbers of the diatonic scale being represented by

$$1,\ \tfrac{9}{8},\ \tfrac{5}{4},\ \tfrac{4}{3},\ \tfrac{3}{2},\ \tfrac{5}{3},\ \tfrac{15}{8},\ 2,$$

if we build a scale on the dominant $\tfrac{3}{2}$, its vibration numbers will be

$$\tfrac{3}{2},\ \tfrac{27}{16},\ \tfrac{15}{8},\ 2,\ \tfrac{9}{4},\ \tfrac{5}{2},\ \tfrac{45}{16},\ 3,$$

or, depressing the last five notes one octave in order to keep to the same part of the key-board,

$$1,\ \tfrac{9}{8},\ \tfrac{5}{4},\ \tfrac{45}{32},\ \tfrac{3}{2},\ \tfrac{27}{16},\ \tfrac{15}{8},\ 2.$$

If we build a scale upon the subdominant $\tfrac{4}{3}$, the vibration numbers will be

$$1,\ \tfrac{10}{9},\ \tfrac{5}{4},\ \tfrac{4}{3},\ \tfrac{3}{2},\ \tfrac{5}{3},\ \tfrac{16}{9},\ 2;$$

where it is to be noted that $\tfrac{4}{3}$ is the tonic, and 1 or 2 the dominant.

More generally, if C, D, E, F, G, A, B, 2C be taken to represent the vibration numbers of the so-named notes in the scale of C, then the vibration numbers of the scales on G and F will be

$$C,\quad D,\quad E,\quad \alpha F,\quad G,\quad \beta A,\quad B,\quad 2C.$$

$$C,\quad \frac{1}{\beta}D,\quad E,\quad F,\quad G,\quad A,\quad \frac{1}{\alpha}B,\quad 2C,$$

where $\alpha = \tfrac{135}{128}$, $\beta = \tfrac{81}{80}$,

α and β being respectively the chromatic semitone and the comma. Here the law of formation of the successive scales is so obvious that they can be written down at sight.

Now let the fifteen scales be written down from C to C♯ in one direction, and to C♭ in the other, when it will be immediately apparent that the symbols are arranged in groups of threes and fours; and if we draw straight lines, horizontal and vertical, so as to enclose each of these groups in a rectangle, we have at once the properly so-called "Natural Fingerboard," the rectangles being the digitals, of which the larger are white, and the smaller coloured. The following table shews one octave of the instrument:—

MAJOR KEYS.								
C′♯	$\alpha\beta$C	$\alpha\beta$D	$\alpha\beta$E	$\alpha\beta$F	$\alpha\beta$G	$\alpha\beta$A	$\alpha\beta$B	2$\alpha\beta$C
F′♯	$\alpha\beta$C	αD	$\alpha\beta$E	$\alpha\beta$F	$\alpha\beta$G	$\alpha\beta$A	βB	2$\alpha\beta$C
B′	$\alpha\beta$C	αD	βE	$\alpha\beta$F	αG	$\alpha\beta$A	βB	2$\alpha\beta$C
E′	αC	αD	βE	$\alpha\beta$F	αG	βA	βB	2αC
A′	αC	D	βE	αF	αG	βA	βB	2αC
D	αC	D	βE	αF	G	βA	B	2αC
G	C	D	E	αF	G	βA	B	2C
C	C	D	E	F	G	A	B	2C
F	C	$\frac{1}{\beta}$D	E	F	G	A	$\frac{1}{\alpha}$B	2C
B♭	C	$\frac{1}{\beta}$D	$\frac{1}{\alpha}$E	F	$\frac{1}{\beta}$G	A	$\frac{1}{\alpha}$B	2C
E♭	$\frac{1}{\beta}$C	$\frac{1}{\beta}$D	$\frac{1}{\alpha}$E	F	$\frac{1}{\beta}$G	$\frac{1}{\alpha}$A	$\frac{1}{\alpha}$B	$\frac{2}{\beta}$C
A♭	$\frac{1}{\beta}$C	$\frac{1}{\alpha\beta}$D	$\frac{1}{\alpha}$E	$\frac{1}{\beta}$F	$\frac{1}{\beta}$G	$\frac{1}{\alpha}$A	$\frac{1}{\alpha}$B	$\frac{2}{\beta}$C
D′♭	$\frac{1}{\beta}$C	$\frac{1}{\alpha\beta}$D	$\frac{1}{\alpha}$E	$\frac{1}{\beta}$F	$\frac{1}{\alpha\beta}$G	$\frac{1}{\alpha}$A	$\frac{1}{\alpha\beta}$B	$\frac{2}{\beta}$C
G′♭	$\frac{1}{\alpha\beta}$C	$\frac{1}{\alpha\beta}$D	$\frac{1}{\alpha\beta}$E	$\frac{1}{\beta}$F	$\frac{1}{\alpha\beta}$G	$\frac{1}{\alpha}$A	$\frac{1}{\alpha\beta}$B	$\frac{2}{\alpha\beta}$C
C′♭	$\frac{1}{\alpha\beta}$C	$\frac{1}{\alpha\beta}$D	$\frac{1}{\alpha\beta}$E	$\frac{1}{\alpha\beta}$F	$\frac{1}{\alpha\beta}$G	$\frac{1}{\alpha\beta}$A	$\frac{1}{\alpha\beta}$B	$\frac{2}{\alpha\beta}$C

Here all the scales are true and adjacent according to their relationship, the fingering being obviously the same for all. Moving from left to right

the interval from a white digital to the adjacent white is 8 : 9, which is also the interval from coloured to coloured. From white to coloured is 9 : 10, and from coloured to white is 15 : 16. The intervals moving in the transverse direction, viz., from the flat keys towards the sharp, express themselves; from white to coloured is α, and from coloured to white is β. In playing any scale, it is easy to remember that the tonic is always a little above the centre of a white digital, and the fifth a little below the centre. The second and seventh are the lower ends of white and coloured digitals respectively. The third is at the centre of a coloured digital. The fourth and sixth are found at the upper ends of white and coloured digitals respectively.

Between digitals related as αG (G♯) and $\frac{1}{\alpha}$A (A♭), that is, between every pair similarly related in mutual azimuth (to borrow a term) and distance, the schisma occurs. The keyboard gives us the value of the schisma by inspection; we may take either the route

$$\frac{1}{\alpha}\text{A to A} = \alpha, \quad \text{A to G} = \frac{9}{10}, \quad \text{G to } \alpha\text{G} = \alpha,$$

giving schisma $= \frac{9}{10}\alpha^2$; or this route,

$$\frac{1}{\alpha}\text{A to A} = \alpha, \quad \text{A to } \beta\text{A} = \beta, \quad \beta\text{A to } \alpha\text{G} = \frac{15}{16},$$

giving schisma $= \frac{15}{16}\alpha\beta$. Or else we may get an entirely numerical value thus:—Since from white to white is 8 : 9, then from $\frac{1}{\alpha\beta}$D diagonally to βA is $(8:9)^4$, and from βA back horizontally to αC, a minor sixth, is 8 : 5, the interval of αC and $\frac{1}{\alpha\beta}$D is, putting Schisma $= \sigma$, we have

$$\sigma = \frac{5}{8}\left(\frac{9}{8}\right)^4.$$

This equation written in the form

$$2^5 . \sigma = \frac{5}{4} . \left(\frac{3}{2}\right)^8$$

proves a somewhat important theorem in temperament; namely, that eight fifths and a major third exceed five octaves by a schisma.

Again, the Comma of Pythagoras, being the excess of twelve fifths over seven octaves, is expressed by

$$\left(\frac{3}{2}\right)^{12} . \frac{1}{2^7} = \left(\frac{9}{8}\right)^6 . \frac{1}{2};$$

that is, it is the excess of six major tones over one octave. The keyboard shows this immediately; from $\frac{1}{\alpha\beta}$D diagonally to $2\alpha\beta$C is six major tones, thence back horizontally to $\alpha\beta$C is one octave; therefore $\alpha\beta$C differs from $\frac{1}{\alpha\beta}$D by the Comma of Pythagoras. And every pair of white digitals—

or coloured, as αD and $\frac{1}{\alpha\beta}$ E—similarly related in azimuth and distance, have the same interval. Obviously, by mere inspection of the board, the Comma of Pythagoras is equal to Comma + Schisma.

So far the major scales only have been considered, without, as yet, any definition of sharps and flats; but it will be necessary to define them before speaking of the minor scales. The following is the notation adopted:—

$$\alpha C = C\sharp, \qquad \beta E = E',$$

$$\alpha\beta C = C'\sharp, \qquad \frac{1}{\alpha} E = E\flat,$$

$$\frac{1}{\beta} C = C^{\backprime}, \qquad \frac{1}{\alpha\beta} E = E^{\backprime}\flat, \text{ and so on.}$$

Proceeding now to the minor scales, I may remark that in its primitive form, the minor scale is a major scale in the order A B C D E F G A; but in modern music the sixth and seventh F and G are sharpened in the ascending scale. Now, to meet this requirement, one note is added in each scale on the "Natural Fingerboard," and it is represented by a round digital in the corner of each coloured digital.

Consider the round digital on A: its value is $\frac{\alpha}{\beta}$ G, that is, it is a chromatic semitone above G‘ and a diatonic semitone below A, and bears to G the relation 25 : 24. It is written G‘ ♯, in conformity with the notation previously established. Thus also the round digital on G‘ is F‘ ♯, that on F‘ is E‘, that on B‘ ♭ is A‘, and so for the others. The relative minor A and its vibration numbers will therefore, in the ascending scale, stand thus:—

A,	B,	C,	D,	E,	F‘ ♯,	G‘ ♯,	A.
1,	$\frac{9}{8}$,	$\frac{6}{5}$,	$\frac{27}{20}$,	$\frac{3}{2}$,	$\frac{5}{2}$,	$\frac{15}{8}$,	2.

According to the views of some theoretical writers, however, the minor scale should differ only in its 6 : 5 third from the major scale. Without offering any opinion as to which form is the most correct, I merely remark that, by playing D‘ instead of D in the above, the vibration numbers become 1, $\frac{9}{8}$, $\frac{6}{5}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{15}{8}$, 2. The tonic minor of C is played thus, in the ascending scale:—

C‘ D‘ E♭ F G‘ A‘ B‘ 2C‘,

or F‘ may be played for F, if preferred: in this case the descending scale would be

C‘, D‘, E♭, F‘, G‘, A♭, B♭, 2C‘.

Having lately had—through the kindness of a friend at Hampstead—an opportunity, in company with Mr. Miller, of seeing and hearing one of Colin Brown's harmoniums, we can certainly say that the instrument appears as successful in practice as it is admirable in theory. The effect of playing C‘ for C in the tonic minor is most pleasing, and has received the distinct approval of eminent musicians.

I have spoken of the facility of playing this instrument. It is, however, only right to qualify this by saying (and it is no disparagement to the instrument) that, with reference to accidentals, it is necessary that the performer should have sufficient knowledge of harmony to understand the key relationship of the piece before him.

The tuning of the Voice-Harmonium is remarkably simple—being by

octaves, perfect fifths, and perfect major thirds. Two only of the thirds are required to be tuned: one to connect the coloured digitals with the white, as E with C, and one to connect the round with the coloured, as G' ♯ with E. No further explanation is required by those who have understood the description of the keyboard just given.

In connection with this subject, I may here explain a very remarkable truth, brought forward by Mr. COLIN BROWN (though Mr. ELLIS states that it has been known before), that all the notes of the scale of C major are harmonics to F, and to F alone. If we multiply the vibration numbers of the scale through by $\frac{3}{4}$, so as to make that of F unity, they will stand as at the bottom of the following table:—

No. of Octave.	Multiplier.	Harmonics with Vibration Numbers.						
5	2^5	C_{24}	D_{27}	E_{30}	F_{32}	G_{36}	A_{40}	B_{45}
4	2^4	C_{12}	.	E_{15}	F_{16}	G_{18}	A_{20}	.
3	2^3	C_6	.	.	F_8	G_9	A_{10}	.
2	2^2	C_3	.	.	F_4	.	A_5	.
1	2	.	.	.	F_2	.	.	.
Diatonic Scale		$\frac{3}{4}$	$\frac{27}{32}$	$\frac{15}{16}$	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$
		C	D	E	F	G	A	B

And a similar table may be formed for each note of the scale. Now, in this, the case of F, the denominators of the fractions representing the other notes are powers of 2. Consequently, when we form the vibration numbers of the octaves above, and pick out as the harmonics of F all the whole numbers, we find that the fifth and successive octaves are all harmonics. Such is not, however, the case with any other note; with C, for instance, F and A can never become harmonics, because their vibration numbers have 3 in the denominator, which never divides 2^n. So in the case of G, only G, B, and D become harmonics. A has only E, A, and B in repeated octaves. E has E and B. B and D have only their own octaves.

Now, as neither 7 nor any higher prime number enters into the ratios of the diatonic scale, it is clear that the vibration numbers of the harmonics formed for the different notes, as in the above table for F, can have among them no primes or multiples of primes greater than 7 inclusive. The series of numbers which excludes these is 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48 ..., and with the solitary exception of 25 all these numbers *are* found among the harmonics. Consider the last nine numbers in the series just written down. They represent the diatonic scale itself, with the addition of a semitone, $\frac{25}{24}$, succeeding the tonic. There is no such note in the scale, but singularly enough it does exist in COLIN BROWN's keyboard; and it is this one addition to the scale, viz., the round digital, which brings in and completes the minor scales. Without wishing to lay too much importance on this circumstance, it cannot but be regarded as very remarkable.

It has been observed that there is a similarity between this keyboard and that of POOLE. There is; yet no one reading the theory given above, and comparing it with ELLIS's remarks on POOLE's (HELMHOLTZ, pp. 534, 677, 678), would say they were in principle the same. COLIN BROWN's is the instrumental embodiment of the Tonic-Sol-fa method of singing,

whereas the characteristic points of POOLE's are the introduction of the harmonic seventh and of what he calls the "double diatonic scale." Notwithstanding all that can be said in favour of the harmonic seventh, the possibility of its general introduction into music cannot be regarded as other than very remote.

[We believe that COLIN BROWN's Voice-Harmonium will be found to be as practically useful as it is here shown to be scientifically and musically correct. The application of the natural fingerboard with perfect intonation to an instrument has naturally led to the enquiry how such an instrument can be played upon and thus become of general use. An answer to this question may be here given in the words of one fully competent to form a sound opinion upon the subject, who, after a thorough examination of the fingerboard, remarked, "that any one conversant with the connection and relationship of keys could master the principles of the fingerboard in half-an-hour; that any player upon the harmonium or organ could become equally familiar with the natural fingerboard by two or three weeks' practice; and that any one wishing to learn to play would save two to three years of time usually spent in mastering the whole round of keys, by learning to play upon this fingerboard, in which the scale in all keys has a similar fingering." The first and second of these statements have already been amply verified, for every one who has tried to play upon this instrument has done so with very little practice. Time will be required to verify the third statement, but that it will be found to be correct can hardly, we think, be a matter of question.]

5156. (By Prof. WOLSTENHOLME, M.A.)—If S_n denote the series

$$\frac{1}{\sin\frac{1}{3}\pi}\left\{\frac{1}{n}\sin n\tfrac{1}{3}\pi+\frac{2n}{1}\cdot\frac{\sin(n-1)\frac{1}{3}\pi}{n-1}+\frac{2n(2n-1)}{1\,.\,2}\cdot\frac{\sin(n-2)\frac{1}{3}\pi}{n-2}+\dots\right\}$$

to n terms; prove (1) that $nS_n=(4n-2)S_{n-1}+3^{n-1}$, and (2) that nS_n is divisible by either 3^n or 3^{n-1}.

Solution by R. TUCKER, M.A.

Let $$S=\frac{\sin n\theta}{n}+2n\frac{\sin(n-1)\theta}{n-1}+\dots+\frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}}\sin\theta,$$

then $$\frac{dS}{d\theta}=\cos n\theta+2n\cos(n-1)\theta+\frac{2n(2n-1)}{1\,.\,2}\cos(n-2)\theta+\dots\dots$$
$$\dots\dots+\frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}}\cos\theta$$
$$=2^{2n-1}\cos^{2n}\tfrac{1}{2}\theta \text{ (Todhunter's \textit{Trigonometry}, Art. 280)};$$

therefore $\quad S = 2^{2n-1} \int \cos^{2n} \tfrac{1}{2}\theta \, d\theta$

$$= 2^{2n-1} \left[\frac{\sin \frac{1}{2}\theta \cos^{2n-1} \frac{1}{2}\theta}{n} + \frac{2n-1}{2n} \int \cos^{2n-2} \tfrac{1}{2}\theta \, d\theta \right] + C;$$

for $\theta = 0$, $S = 0$, therefore $C = 0$ [for $\int \cos^{2n} \frac{1}{2}\theta \, d\theta$ ultimately becomes $d\theta$, that is, θ]; hence we have

$$S_n = \frac{2^{2n-1}}{\sin \frac{1}{3}\pi} \left[\frac{\sin \frac{1}{6}\pi \cos^{2n-1} \frac{1}{6}\pi}{n} + \frac{2n-1}{2n} \int \cos^{2n-2} \tfrac{1}{2}\theta \, d\theta \right];$$

therefore

$$n S_n - (4n-2) S_{n-1} = 2^{2n-1} \cdot \frac{2}{\sqrt{3}} \left[\frac{1}{2} \cdot \left(\frac{\sqrt{3}}{2} \right)^{2n-1} \right] = (\sqrt{3})^{2n-2} = 3^{n-1}.$$

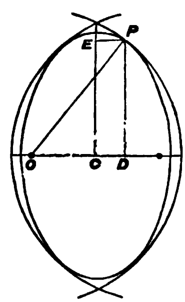
$$S_1 = 1, \quad 2S_2 = 6S_1 + 3 = 9 = 3^2,$$

$$3S_3 = 10S_2 + 3^2 = 54 = 3^3 (2), \quad 4S_4 = 14S_3 + 3^3 = 3^2(3+28);$$

so that (2) does not appear to hold.

4962. (By E. B. Spitz.)—If a circle be placed at random on another equal circle, prove that the average area of the greatest ellipse that can be inscribed in the area common to the two circles is, in parts of the area of one of the circles, $\frac{1}{2}\pi - \frac{4}{3}$.

Solution by Artemas Martin.



Let a and b be the semi-axes of the maximum ellipse. Take the origin at the centre of the ellipse, and put $PD = x$, $CD = EP = y$; then its equation is

$$a^2y^2 + b^2x^2 = a^2b^2 \quad\text{..................(1).}$$

Let $OC = z$, then $OD = y + z$, and the equation to the circle (O) is $\quad (y+z)^2 + x^2 = r^2$.........(2).

Also, $\pi ab =$ maximum, or $a^2b^2 =$ maximum ...(3).

If Δ = the average area required, we have

$$\Delta = \frac{\int_0^r \pi ab \cdot 4\pi z \, dz}{\int_0^r 4\pi z \, dz} = \frac{2\pi}{r^2} \int_0^r abz \, dz.$$

The circle and ellipse have a common tangent at P; therefore, differentiating (1) and (2) with reference to x and y, we obtain

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} = -\frac{x}{y+z} \quad\text{..............................(4);}$$

whence $y = \dfrac{b^2 z}{a^2 - b^2}$; therefore $y + z = \dfrac{a^2 z}{a^2 - b^2}$.

From (1) we have $x^2 = a^2 - \dfrac{a^2 b^2 z^2}{(a^2 - b^2)^2}$.

Substituting in (3), we have $b^2 = a^2 - \dfrac{a^2 z^2}{r^2 - a^2}$ ....(5);

therefore $a^2 b^2 = a^4 - \dfrac{a^4 z^2}{r^2 - a^2} = \text{maximum} = u$ ....(6).

Differentiating, $\dfrac{du}{da} = 4a^3 - \dfrac{4a^3 r^2 z^2 - 2a^5 z^2}{(r^2 - a^2)^2} = 0$;

whence we obtain $z^2 = \dfrac{2(r^2 - a^2)^2}{2r^2 - a^2}$ ....(7).

Substituting in (6), we have $ab = \dfrac{a^3}{(2r^2 - a^2)^{\frac{1}{2}}}$ ....(8).

Differentiating (7), $z\,dz = \dfrac{2r^4 a\,da}{(2r^2 - a^2)^2} - 2a\,da$ ....(9).

Therefore
$$\Delta = \frac{4\pi}{r^2} \int_0^r \frac{a^4\,da}{(2r^2 - a^2)^{\frac{1}{2}}} - 4r^2\pi \int_0^r \frac{a^4\,da}{(2r^2 - a^2)^2 (2r^2 - a^2)^{\frac{1}{2}}}$$
$$= r^2\pi\left(\tfrac{3}{2}\pi - 4\right) - 4r^2\pi \int_0^r \frac{a^4\,da}{(2r^2 - a^2)^2 (2r^2 - a^2)^{\frac{1}{2}}}.$$

Let $2r^2 - a^2 = w^2$; then $a = (2r^2 - w^2)^{\frac{1}{2}}$, $da = \dfrac{-w\,dw}{(2r^2 - w^2)^{\frac{1}{2}}}$,

$$\Delta = r^2\pi\left(\tfrac{3}{2}\pi - 4\right) - 4r^2\pi \int_r^{r\sqrt{2}} \frac{(2r^2 - w^2)^2\,dw}{w^4 (2r^2 - w^2)^{\frac{1}{2}}}$$
$$= r^2\pi\left(\tfrac{3}{2}\pi - 4\right) - 16r^6\pi \int_r^{r\sqrt{2}} \frac{dw}{w^4 (2r^2 - w^2)^{\frac{1}{2}}} + 16r^4\pi \int^{r\sqrt{2}} \frac{dw}{w^2 (2r^2 - w^2)^{\frac{1}{2}}}$$
$$- 4r^2\pi \int_r^{r\sqrt{2}} \frac{dw}{(2r^2 - w^2)^{\frac{1}{2}}} = r^2\pi\left(\tfrac{1}{2}\pi - \tfrac{4}{3}\right).$$

5151. (By the Rev. H. T. Sharpe, M.A.)—One end of a heavy rod rests on a horizontal plane and against the foot of a vertical wall, the other end rests against a parallel vertical wall, all the surfaces being smooth. Show that, if it slips down, the angle ϕ through which it turns about the common normal to the vertical walls is given by the equation

$$(1 + 3\sin^2\phi)\left(\frac{d\phi}{dt}\right)^2 = \frac{6g}{(a^2 - b^2)^{\frac{1}{2}}}(1 - \cos\phi),$$

where $2a$ is the length of the rod, and $2b$ the distance between the walls.

I. *Solution by* J. W. SHARPE, B.A.

The centre of gravity of the rod must remain throughout the motion in the same vertical plane perpendicular to the two walls. Take this plane for the plane of xOz, the line along which the lower end of the rod moves for Oy, and the common normal to the two walls for Oz, and take Ox vertically upwards.

The coordinates of the centre of gravity will be $(x, 0, a\cos\theta)$, where θ is the inclination of the rod to Oz, and therefore is constant.

Let (ξ, η, ζ) be the coordinates of any point of the rod relatively to its centre, at a distance ρ from it, and let m denote the mass of the rod; then the semi *vis viva* of the rod, at any time t, is

$$V = \int_0^a \left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right\} m\,\frac{d\rho}{2a} + \frac{m}{2}\left(\frac{dx}{dt}\right)^2, \text{ for } \frac{d\zeta}{dt} = 0.$$

Now $$\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 = \rho^2 \sin^2\theta \left(\frac{d\phi}{dt}\right)^2, \ \theta \text{ being constant};$$

$$\left(\frac{dx}{dt}\right)^2 = a^2 \sin^2\theta \sin^2\phi \left(\frac{d\phi}{dt}\right)^2;$$

therefore $$V = \frac{ma^2}{6}\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{ma^2}{2}\sin^2\theta\sin^2\phi\left(\frac{d\phi}{dt}\right)^2;$$

and the work done by gravity is

$$mg\left[(a^2-b^2)^{\frac{1}{2}} - a\sin\theta\cos\phi\right];$$

and the work done by the reactions of the smooth planes is nothing; therefore the equation of *vis viva* is that given in the Question.

II. *Solution by* S. TEBAY, B.A.

Let the plane xz pass through the centre of gravity of the rod, and let (x, y, z) be any point in it at the distance r from the centre; then, putting $a^2 - b^2 = h^2$, we find

$$x = \frac{b}{a}(a-r), \quad y = \frac{h}{a}\,r\cos\phi, \quad z = \frac{h}{a}(a+r)\sin\phi.$$

By the principle of *vis viva*, we have

$$\int dr\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \text{const.} - 2g\int z\,dr,$$

which immediately gives

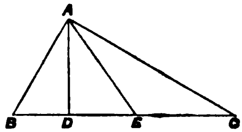
$$(1 + 3\cos^2\phi)\left(\frac{d\phi}{dt}\right)^2 = \frac{6g}{h}(\sin\phi' - \sin\phi);$$

ϕ' being the initial value of ϕ.

5145. (By R. TUCKER, M.A.)—Lines are drawn from the vertex A of a triangle to the base, so that the angles BAD, DAE, EAC are equal; r, ρ, r', ρ' are the inscribed and escribed (to side AC) radii of the triangles AEC, ABC; prove that, if AD is perpendicular to BC, then

$$\frac{\rho r' + \rho' r}{\rho r' - \rho' r} = 2\cos\frac{A}{3}.$$

Solution by R. E. Riley, B.A.; R. Battle; *and many others.*



We have $B = 90^\circ - \frac{1}{3}A, \quad C = 90^\circ - \frac{2}{3}A;$

$$r = b \sin \tfrac{1}{6}A \sin \tfrac{1}{2}C \operatorname{cosec} \tfrac{1}{2}B,$$
$$\rho = b \cos \tfrac{1}{6}A \cos \tfrac{1}{2}C \operatorname{cosec} \tfrac{1}{2}B,$$
$$r' = b \sin \tfrac{1}{2}A \sin \tfrac{1}{2}C \sec \tfrac{1}{2}B,$$
$$\rho' = b \cos \tfrac{1}{2}A \cos \tfrac{1}{2}C \sec \tfrac{1}{2}B;$$

therefore $$\frac{\rho r' + \rho' r}{\rho r' - \rho' r} = \frac{\sin \frac{2}{3}A}{\sin \frac{1}{3}A} = 2 \cos \tfrac{1}{3}A.$$

5078. (By R. Tucker, M.A.)—In a rectangular hyperbola $(2xy = a^2)$, if m be the tangent of the inclination of one of the normals through (x, y) (this being a point from which four normals are drawn), prove (1) that

$$\Sigma(m) = \frac{2x^2}{a^2}, \quad \Sigma\left(\frac{1}{m}\right) = \frac{2y^2}{a^2}.$$

Also shew (2) that $\Sigma(CP^2) = D^2$, where C is the centre, P one of the points where a circle cuts the hyperbola, and D the diameter of the circle; and hence (3) find the distance of the point where the circle of curvature cuts the curve again from the centre. Also (4) if chords are drawn through a focus, find the lengths of the chords, and prove that their centres lie upon a hyperbola through C.

Solution by the Proposer; R. E. Riley, B.A.; *and others.*

The equation to the hyperbola is $2xy = a^2$ .... (1);
hence the equation to the normal is

$$yy' - xx' = y'^2 - x'^2, \quad \text{or} \quad y = mx + \frac{a}{\sqrt{2m}} - \frac{am^{\frac{3}{2}}}{\sqrt{2}}.$$

Clearing of radicals, we get

$$a^2m^4 - 2m^3x^2 - 2m^2(a^2 - 2xy) - 2my^2 + a^2 = 0,$$

whence $$\Sigma(m) = \frac{2x^2}{a^2}, \quad \Sigma\left(\frac{1}{m}\right) = \frac{2y^2}{a^2}.$$

To find where the circle $x^2 + y^2 + Ax + By + C = 0$ cuts the curve, we must solve the equation

$$x^4 + Ax^3 + Cx^2 + \tfrac{1}{2}(a^2B)x + \tfrac{1}{4}a^4 = 0;$$

whence $$\Sigma x^2 = A^2 - 2C, \quad \Sigma y^2 = B^2 - 2C,$$

and $$\Sigma(CP)^2 = \Sigma(x^2 + y^2) = A^2 + B^2 - 4C = \text{square on diameter};$$

hence (3) $$3CP^2 + CQ^2 = D^2 = 4\frac{CP^6}{a^4},$$

whence CQ is known in terms of CP.

The equation to a focal chord is $y-a = m(n-a)$; whence

$$x^2 - \frac{m-1}{m}ax - \frac{a^2}{2m} = 0, \quad y^2 + (m-1)\,ay - \tfrac{1}{2}ma^2 = 0;$$

therefore $\delta^2 = (x_1-x_2)^2 + (y_1-y_2)^2 = \dfrac{(m^2+1)^2}{m^2}a^2$, and $\delta = \dfrac{m^2+1}{m}a$.

To find the locus of centres we have the equations

$$2X = \frac{m-1}{m}a, \quad 2Y = -(m-1)\,a;$$

whence $$\frac{1}{X} + \frac{1}{Y} = \frac{2}{a}, \quad \text{or} \quad 2XY = a(X+Y),$$

another rectangular hyperbola with its vertices at the centre and focus of the given hyperbola, and coincident transverse axis.

5131. (By Prof. SYLVESTER, F.R.S.)—When a pencil of rays falling on a curved surface are each *deflected through a constant angle*, the curve which the deflected rays envelop may be called the Caustic of Deflexion. Suppose now that a pencil of rays issuing from a fixed point fall upon a circle; prove (1) that the caustic of deflexion is a conic, find its equation in rectangular coordinates, and show that its discriminant is a perfect square. Prove also (2) that when the centre of the incident pencil approaches indefinitely near to the circle, the limiting form of the caustic of deflexion is a pair of coincident straight lines; and again, that when the angle of deflexion approaches indefinitely near to zero, the limiting form of the caustic is a pair of crossing straight lines, which are real if the centre of the pencil is inside, and imaginary if it is outside the circle; the ultimate as distinguished from the limiting forms of the caustic in each of the cases supposed being evidently a single point. Explain also what happens when the tangent of the angle of deflexion is $\sqrt{(-1)}$.

I. *Solution by* Prof. TANNER, M.A.; S. FORDE, M.A.; *and others.*

Take the centre of given circle as origin; its radius as unit of length; the axis of x through the centre of the pencil of rays; and the distance of this point from the origin, λ; then the equation of the ray from λ, 0 to x, y is

$$\eta - y = \frac{y}{x-\lambda}(\xi - x).$$

If the result of deflection be to increase by α the inclination of the ray to axis of x, the equation of deflected ray will be

$$(\eta-y)(x-\lambda-y\tan\alpha)-(\xi-x)(y+x\tan\alpha-\lambda\tan\alpha) = 0,$$

or $\xi\lambda\tan\alpha - \eta\lambda + \tan\alpha + x(\eta - \xi\tan\alpha - \lambda\tan\alpha) + y(\lambda - \xi - \eta\tan\alpha) = 0$,

if we suppose x, y to lie on the circle $x^2+y^2 = 1$.

Taking the envelope of this line, we get the equation of the caustic of deflection in the form

$$(\xi\lambda\tan\alpha - \eta\lambda + \tan\alpha)^2 = (\eta - \xi\tan\alpha - \lambda\tan\alpha)^2 + (\lambda - \xi - \eta\tan\alpha)^2 \ldots\ldots (1),$$

or $\xi^2(\lambda^2\tan^2\alpha-\sec^2\alpha)-2\xi\eta\lambda^2\tan\alpha+\eta^2(\lambda^2-\sec^2\alpha)$

$$+2\xi\lambda+2\eta\lambda\tan\alpha+\tan^2\alpha-\lambda^2\sec^2\alpha=0\ \ldots\ldots\ldots(2).$$

The discriminant of (2) becomes $(\lambda^2-1)^2\tan^2\alpha.\sec^4\alpha$, and is therefore a perfect square, vanishing when $\lambda = 1$, when $\alpha = 0$, when $\sec\alpha = 0$, (or what is equivalent, when $\tan\alpha = \sqrt{-1}$).

When $\lambda = 1$, or the centre of the pencil is on circumference, (1) becomes $(1-\xi-\eta\tan\alpha)^2 = 0$, which represents two coincident lines. When $\alpha = 0$, (1) becomes $\xi^2-(\lambda^2-1)\eta^2 = 0$, which represents two lines, possible or impossible, as λ is greater or less than 1; viz., as the radiant point is outside or inside the circle. When $\tan^2\alpha = -1$, and therefore $\sec^2\alpha = 0$, (2) gives only the point $(\lambda^{-1}, 0)$.

II. *Solution by the* Rev. John George Birch, M.A.; Professor Evans, M.A.; Professor Nash, M.A.; *and others.*

Take for axis of x the line through the centre of the circle, making an angle equal to the angle of deflection with the line passing through the centre of the circle and the centre of the incident pencil; and for axis of y the perpendicular to this through the centre of the pencil. Let a = distance between the centre of the circle and the centre of the pencil, r = radius of circle, α = angle of deflection. Put also θ = angle which an incident ray makes with the line joining the centre of the circle and the centre of the pencil; and ϕ = the angle which the radius of the circle, which passes through the point in which the incident ray meets the circle, makes with the line joining the centre of pencil and centre of circle.

Then the deflected ray makes an angle = θ with the axis of x; its equation must therefore be of the form $y = \tan\theta.x+k$.

Determining k by making this line pass through the point in which the incident ray meets the circle, we get for the equation to the deflected ray

$$y-r\sin(\phi-\alpha) = \tan\theta\left\{x-a\cos\alpha-r\cos(\phi-\alpha)\right\}.$$

But as $\dfrac{r}{a} = \dfrac{\sin\theta}{\sin(\phi+}$, and therefore $\tan\theta = \dfrac{r\sin\phi}{a-r\cos\phi}$,

the equation to the deflected ray becomes

$$y-r\sin(\phi-\alpha) = \frac{r\sin\phi}{a-r\cos\phi}\left\{x-a\cos\alpha-r\cos(\phi-\alpha)\right\}$$

or $$ay+ar\cos\phi\sin\alpha-r(y\cos\phi+x\sin\phi)-r^2\sin\alpha = 0\ \ldots\ldots\ldots\ (1).$$

Differentiating with regard to ϕ, we have

$$(y-a\sin\alpha)\sin\phi = x\cos\phi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

By eliminating ϕ between (1) and (2), we find for the caustic of the deflected rays $$(a^2-r^2)\,y^2-r^2x^2 = r^2(a^2-r^2)\sin^2\alpha\ \ldots\ldots\ldots\ldots\ldots\ (3),$$

an hyperbola whose vertices are situated on the axis of y at the distance $r\sin\alpha$ from the origin. When a decreases, the asymptotes of this hyperbola close in upon the axis of y. When a is very nearly equal to r, they are inclined at a very small angle to the axis of y. When $a=r$, the ultimate form of the hyperbola is coincident lines proceeding from two points in the axis of y, situated one on each side of the origin at the distance $r\sin\alpha$ from it, to infinity, not of course including the space $r\sin\alpha$ at each side of the origin. And then all the deflected rays pass through one or other of the points in which the axis of y cuts the circle.

When a remains constant, and α decreases, the asymptotes of the hyperbola retain the same position, while its vertices move in towards the origin; and when actually $\alpha = 0$, the ultimate form of the hyperbola is the asymptote. All the deflected rays in this case pass through the origin, which then coincides with centre of incident pencil.

When a becomes less than r, the caustic assumes form of an ellipse. If at the same time $\alpha = 0$, the locus becomes $(r^2 - a^2)\,y^2 + r^2 x^2 = 0$, which is satisfied only for real values of x and y by origin.

5072. (By J. L. McKenzie, B.A.)—Given any number of equations $\phi(x) = 0$, $\psi(y) = 0$, ...; show that it is possible, in most cases, to express in the form of a determinant the equation whose roots are of the form $f(x, y, \ldots)$, where f is any algebraic function.

Solution by the Proposer.

Let $\alpha_r, \beta_r, \ldots$ be the sums of the r^{th} powers of the roots of the given equations, and s_r of the required equation. Then

$$s_r = \left\{f(a_1 b_1 \ldots)\right\}^r + \left\{f(a_2 b_1 \ldots)\right\}^r + \ldots\ldots$$
$$+ \left\{f(a_1 b_2 \ldots)\right\}^r + \left\{f(a_2 b_2 \ldots)\right\}^r + \ldots\ldots \text{ \&c.}$$

Thus to any term $\lambda x^m y^n \ldots$ in the expansion of $\{f(x, y, \ldots)\}^r$ there will correspond in s_r the sum

$$\lambda a_1^m b_1^n + \lambda a_2^m b_1^n \ldots + \ldots + \lambda a_1^m b_2^n + \lambda a_2^m b_2^n \ldots + \ldots \text{ \&c.} = \lambda \alpha_m \beta_n \ldots.$$

Hence s_r may be found by expanding $\{f(x, y, \ldots)\}^r$, and substituting $\alpha_m, \beta_n \ldots$ for $x^m, y^n \ldots$ in every term of the expansion. This evidently applies whether $m, n, \ldots$ be positive or negative, integral or fractional, so long as $\{f(x, y, \ldots)\}^r$ can be expanded in a finite number of terms. If $f(x, y, \ldots)$ contain an expression of the form $(\lambda x^n + \mu x^{n-1} \ldots)^m$, where m is negative or fractional, the terms in s_r involving this expression may be found by making the transformation $x' = \lambda x^n + \mu x^{n-1} \ldots$ in $\phi(x)$. And if $\{f(x, y, \ldots)\}^r$ contain a term of the form $\{\mathrm{F}(x, y, \ldots)\}^m$, where m is negative or fractional, form the equation whose roots are $\mathrm{F}(x, y, \ldots)$; then the term in s_r corresponding to $\{\mathrm{F}(x, y, \ldots)\}^m$ will be found as the sum of the m^{th} powers of the roots of this equation. I believe the only case in which this method is not applicable, is when $\{f(x, y, \ldots)\}^r$ contains a fractional term, the denominator of the fraction containing more than one term, and some of the variables being involved in both the numerator and denominator. If the denominator involves one set of variables and the numerator an entirely different set, the method still applies. For example, suppose the fractional term to be $\lambda x^m (\mu y^p + \nu z^q)^{-n}$; then, for any value a_1 of x, we will have $\lambda a_1^m \cdot \Sigma (\mu b^p + \nu c^q)^{-n}$. Thus the term in s_r will be $\lambda \alpha_m \Sigma (\mu b^p + \nu c^q)^{-n}$; and $\Sigma (\mu b^p + \nu c^q)^{-n}$ can be found, as the sum of the $(-n)^{\text{th}}$ powers of the roots of the equation whose roots are of the form $\mu y^p + \nu z^q$.

Thus s_r may be found; in practice, however, the calculations required, even in the simplest cases, are very laborious.

When s_r is known, the required equation is obtained in the form of a determinant by eliminating $1, p_1, p_2, \dots p_n$ between

$$u^n + p_1 u^{n-1} + p_2 u^{n-2} + \dots p_n = 0,$$

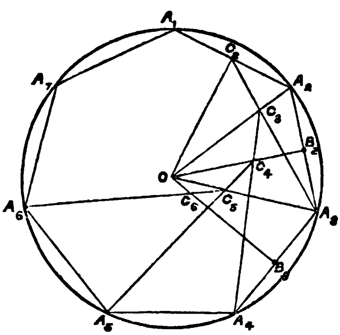
and any n of the equations

$$s_1 + p_1 = 0, \quad s_2 + p_1 s_1 + 2p_2 = 0, \quad \&c.$$

The value of n must previously be found by considering the number of roots which the required equation ought to have; for example, if $f(x, y, \dots)$ is rational, n is equal to the product of the degrees of $\phi(x)$, $\psi(y)$, &c.

4186. (By the Editor.)—Let $A_1, A_2, A_3 \dots A_n$ be the vertices of a regular polygon inscribed in a circle whose centre is O; $C_2, B_2, B_3, B_4 \dots \dots B_n$ the feet of the perpendiculars from O upon the sides A_1A_2, A_2A_3, $A_3A_4 \dots A_{n-1}A_n$; then, if C_2A_3 cut OA_2 in C_3; C_3A_4 cut OB_2 in C_4; C_4A_5 cut OA_3 in C_5; C_5A_6 cut OB_3 in C_6, &c. &c.; prove that $C_2C_3 : C_3A_3 = 1 : 2$, $C_3C_4 : C_4A_4 = 1 : 3$, ..., $C_{n-1}C_n : C_nA_n = 1 : n-1$; C_n being coincident with O.

I. *Solution by* Prof. Evans, M.A.; Christine Ladd; J. L. McKenzie; Elizabeth Blackwood; *and others.*



The areas of the triangles $OC_{m-1}C_m$ and $OC_{m-1}A_m$ (where m may have any of the values 3, 4, 5 ... $n-2$), are in the ratio $C_{m-1}C_m : C_{m-1}A_m$; hence we have

$$\frac{C_{m-1}C_m}{C_{m-1}A_m} = \frac{OC_{m-1} \,.\, OC_{m+1} \sin\frac{\pi}{n}}{OC_{m-1} \,.\, r \sin\frac{m\pi}{n}} \quad (1),$$

$$\frac{C_m C_{m+1}}{C_{m+1}A_m} = \frac{OC_m \,.\, OC_{m+1} \sin\frac{\pi}{n}}{OC_{m+1} \,.\, r \sin\frac{m\pi}{n}} \quad (2);$$

therefore $C_mC_{m+1} : C_{m+1}A_m$

$$= C_{m-1}C_m : C_{m-1}A_m.$$

But when $m = 3$, $C_2C_3 : C_2A_3 = 1 : 3$, because C_2A_3 and OA_2 are median lines of the triangle $A_1A_2A_3$; hence generally, by induction,

$$C_mC_{m+1} : C_{m+1}A_{m+1} = 1 : m \quad \text{.........................(3).}$$

When $m = n-1$, equation (2) fails, because the triangle $OC_{n-1}A_n$ vanishes; but we have

$$\frac{OC_{n-1}}{OA_n} = \frac{OC_{n-2} \,.\, OC_{n-1} \sin\frac{\pi}{n}}{OC_{n-2} \,.\, r \sin\frac{(n-1)\pi}{n}} = \frac{C_{n-2}C_{n-1}}{C_{n-2}A_{n-1}} = \frac{1}{n-1}, \text{ by (3).}$$

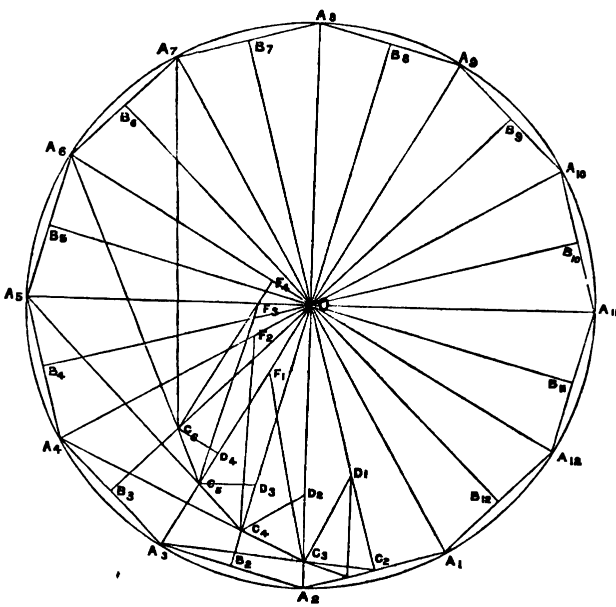
[By mechanico-geometrical considerations a proof may be readily obtained as follows :—

Let $m_1, m_2, m_3 \dots m_n$ denote a system of equal material particles located at $A_1, A_2, A_3 \dots A_n$ respectively, the weight of each being m. The common centre of gravity of two, three, four, five, six, &c. of these particles is evidently in OC_2, OA_2, OB_2, OA_3, OB_3, &c. respectively; therefore, since C_2 is the common centre of gravity of m_1 and m_2, C_3 is that of m_1, m_2, m_3; C_4 that of m_1, m_2, m_3, m_4; C_5 that of m_1, m_2, m_3, m_4, m_5, &c.; C_n the centre of gravity of the whole system, coinciding with O. Since $2m \,.\, C_2C_3 = m \,.\, C_3A_3$, $3m \,.\, C_3C_4 = m \,.\, C_4A_4$, $4m \,.\, C_4C_5 = m \,.\, C_5A_5$, and $(n-1)\, m \,.\, C_{n-1}C_n = m \,.\, C_nA_n$, we have

$$C_2C_3 : C_3A_3 = 1 : 2, \quad C_3C_4 : C_4A_4 = 1 : 3, \quad C_{n-1}C_n : C_nA_n = 1 : n-1.]$$

II. *Solution by* S. A. Renshaw; E. Rutter; *and others.*

Take OA_3, OC_2 as axes of X and Y, and call the $\angle A_2OC_2 = \phi$; then



the equations of A_3C_2, OC_3 are

$$\frac{x}{r} + \frac{y}{r\cos\phi} = 1, \quad y = \frac{x\sin 2\phi}{\sin\phi} = 2x\cos\phi \;\dots\dots\dots\dots\; (1, 2);$$

therefore, at C_3, $3x = r$; hence $C_2C_3 = \frac{1}{3}\, C_2A_3$. Again, $OC_3 = \dfrac{r\sin 3\phi}{\sin\phi}$;

hence, taking as axes of X and Y, OA_4, OC_3 respectively, the equations of

A_4C_3, OC_4 are $$\frac{x}{r}+\frac{3y\sin\phi}{r\sin 3\phi}=1,\quad y=\frac{x\sin 3\phi}{\sin\phi}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(3,4);$$

therefore, at C_4, $4x=r$, hence $C_3C_4=\frac{1}{4}C_3A_4$; and so on. And it is evident that the line $C_{n-1}D_{n-3}$, drawn from C_{n-1} parallel to OA_{n-1}, will make with the line OC_{n-2} an angle $=(n-1)\phi$, and that

$$C_{n-1}D_{n-3}=\frac{r}{n-1};$$

hence $$\frac{C_{n-1}O}{C_{n-1}D_{n-3}}=\frac{\sin(n-1)\phi}{\sin\phi},$$

therefore $$C_{n-1}O=\frac{r}{n-1}\cdot\frac{\sin(n-1)\phi}{\sin\phi}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(a).$$

Taking therefore as axes of X and Y, OA_{n-1}, OC_{n-2}, the equations of $C_{n-2}A_{n-1}$, $C_{n-1}O$ are

$$\frac{x}{r}+\frac{y(n-1)\sin\phi}{r\sin(n-1)\phi}=1,\quad y=\frac{x\sin(n-1)\phi}{\sin\phi}\;\ldots\ldots\ldots\;(5,6).$$

Therefore, at C_{n-1}, $nx=r$, hence $C_{n-1}D_{n-3}=\frac{r}{n}$,

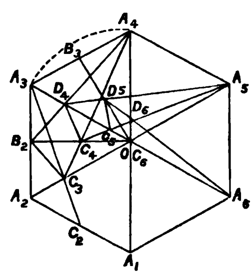
therefore $$C_{n-2}C_{n-1}=\frac{C_{n-2}A_{n-1}}{n}.$$

If therefore the theorem is true for $(n-1)$ it is also true for n. But it has been proved above to be true for 3 and 4; it is therefore true generally.

Further, it is evident that $C_nO=\frac{r}{n}\cdot\frac{\sin n\phi}{\sin\phi}$ [see (a) above]; and, when $n\phi=360$, this vanishes, or C_n coincides with O.

A similar theorem will also hold for the circumscribed polygon.

III. *Solution by* ELIZABETH BLACKWOOD; C. LEUDESDORF, M.A.; *and others.*



Suppose the polygon to be a hexagon, as in the annexed diagram, and having drawn the lines stated in the enunciation of the question, draw also B_2A_4 cutting OA_3 in D_4; D_4A_5 cutting OB_3 in D_5; D_5A_6 cutting OA_4 in D_6.

Now $C_2C_3=B_2D_4$; $C_3A_3=D_4A_4$; $C_3C_4=D_4D_5$; $C_4A_4=D_5A_5$; $C_4C_5=D_5D_6$; $C_5A_5=D_6A_6$; $C_5C_6=D_6C_6$, and C_6 evidently coincides with O. It is also manifest that the triangle $A_2C_2C_3=A_2B_2C_3=A_3B_2C_3$, therefore $\Delta A_3A_2C_3=2A_2C_2C_3$. Consequently

$$C_2C_3:C_3A_3=\Delta A_2C_2C_3:A_3A_2C_3=1:2\ldots(a).$$

Now, as $C_2C_3=B_2D_4$, and $C_3A_3=D_4A_4$, therefore $D_4A_4=2B_2D_4$, or

$B_2A_4 = 3B_2D_4$, therefore the triangle $A_4B_2C_4 = 3\,D_4B_2C_4 = 3B_2C_3C_4$, the triangles $D_4B_2C_4$, $B_2C_3C_4$ being equal in all respects.

Therefore $$C_3C_4 : C_4A_4 = \Delta\, B_2C_3C_4 : A_4B_2C_4 = 1 : 3 \;\;\ldots\ldots\ldots\ldots \;\; (b).$$

Again, as $C_3C_4 = D_4D_5$, and $C_4A_4 = D_5A_5$, therefore $D_5A_5 = 3D_4D_5$, or $D_4A_5 = 4D_4D_5$; therefore the triangle $A_6D_4C_5 = 4\,D_5D_4C_5 = 4D_4C_4C_5$, the triangles $D_5D_4C_5$, $D_4C_4C_5$ being equal in all respects.

Therefore $$C_4C_5 : C_5A_5 = \Delta\, D_4C_4C_5 : A_5D_4C_5 = 1 : 4 \;\;\ldots\ldots\ldots\ldots \;\; (c).$$

Similarly it may be proved that

$$C_5C_6 : C_6A_6 = \Delta\, D_5C_5C_6 : A_6D_5C_6 = 1 : 5 \;\;\ldots\ldots\ldots\ldots \;\; (d).$$

Therefore, collecting from (*a*), (*b*), (*c*), (*d*), we get

$$\begin{aligned} C_2\,C_3 : C_3\,A_3 &= 1 : 2, \\ C_3\,C_4 : C_4\,A_4 &= 1 : 3, \\ C_4\,C_5 : C_5\,A_5 &= 1 : 4, \\ C_5\,C_6 : C_6\,A_6 &= 1 : 5. \end{aligned}$$

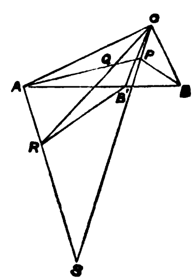
And it is obvious from what precedes that in a polygon of n vertices we should ultimately obtain

$$C_{n-1}\,C_n : C_n\,A_n = 1 : n-1; \;\; C_n \text{ being coincident with O.}$$

When the polygon has an *odd* number of vertices instead of an *even* number, the method of solution is still the same.

5122. (By R. Tucker, M.A.)—ABC, AB′C are two triangles satisfying the ambiguous case; P, Q, R, S are the centres of the inscribed circles and of the circles escribed to the variable side (c); prove that a circle will go round PQRS.

I. *Solution by* R. Battle, M.A.; E. Rutter; S. A. Renshaw; *and many others.*



Because the angles B and CB′B are equal and bisected by BP, RB′, we have $\angle ABP = AB'R$. Again, the points A, S, B, P are in the circumference of a circle, and so are the points A, R, B′, Q; therefore $\angle ABP = ASP$, and $\angle AQR = AB'R$; whence $\angle AQR = ASP$; consequently the points P, Q, R, S are in the circumference of a circle.

II. *Solution by* Prof. Wolstenholme; Prof. Evans; *and others.*

The points P, Q lie on the internal bisector of the angle A, and R, S on the external bisector, and if A, B, C; A, B′, C be the angles of the two triangles, $R = \frac{1}{2}a \operatorname{cosec} A$ the radius of the circle ABC or AB′C,

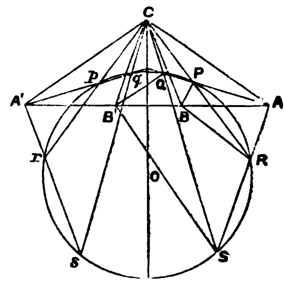
$AP = 4R \sin \frac{1}{2}B \sin \frac{1}{2}C$, $AQ = 4R \sin \frac{1}{2}B' \sin \frac{1}{2}C'$, $AR = 4R \cos \frac{1}{2}B \sin C$, $AS = 4R \cos \frac{1}{2}B' \sin \frac{1}{2}C'$; also $B + B' = 180°$, so that $\sin \frac{1}{2}B' = \cos \frac{1}{2}B$, and $\cos \frac{1}{2}B' = \sin \frac{1}{2}B$, whence we have

$$AP \cdot AQ = AR \cdot AS = 4R^2 \sin B \left[\cos \tfrac{1}{2}(C - C') - \cos \tfrac{1}{2}(C + C')\right]$$
$$= 4R^2 \sin B (\sin B - \sin A) = b^2 - ab,$$

or P, Q, R, S lie on a circle.

The square of the radius of this circle will be found to be $\frac{a^2}{\sin^2 A} - ab$.

III. *Solution by* S. A. Renshaw; A. Martin; *and others.*



Plainly, the angles AB′S, PBA are equal, being the halves of equal angles; that is, since circles go round PBRA and QB′SA, the angle AQS = PRA, therefore P, Q, R, S are in a circle.

If AB′ be produced to A′, so that B′A′ = AB, and the symmetrical figure be completed, it is evident in the same way that p, q, r, s are also in a circle. Now it may easily be shown that CP = Cp and CR = Cr; a circle may therefore be drawn through P, p, R, r, *and having its centre on the bisector of the angle* BCB′. But since AC, AQ, AB, AS, and A′C, A′q, A′r, A′s are evidently harmonic pencils, therefore AA′ is the polar of C with regard to the circle PpRr, and consequently this circle must also pass through Q, q, S, s. That is, the eight centres of inscribed and escribed circles, taken as directed, lie on a circle of which the centre is situated upon the internal bisector of the angle BCB′.

5103. (By R. F. Davis, B.A.)—A heavy particle slides down a rough parabolic arc whose axis is vertical and concavity upwards. Prove that the pressure on the curve is a maximum when the direction of motion makes with the horizontal an angle whose tangent is $\frac{1}{2}\mu$.

Solution by the Proposer; R. E. Riley, B.A.; *and others.*

Measuring ϕ from the tangent at the vertex, and s in the direction of motion, the equations determining the motion will be

$$mv\frac{dv}{ds} = mg \sin \phi - \mu R, \text{ and } \frac{mv^2}{\rho} = R - mg \cos \cos \phi \ldots\ldots (1, 2).$$

Now $$\rho = -\frac{ds}{d\phi} = 2a \sec^3 \phi;$$

therefore, eliminating R and arranging, we have

$$\frac{av^3}{d\phi} - 2\mu v^2 = 4ag\,(\mu \sec^2\phi - \sec^3\phi \sin\phi).$$

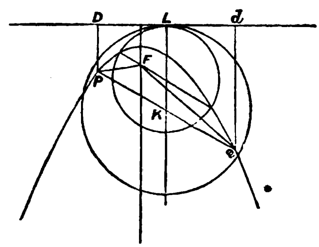
This equation, when integrated, becomes

$$v^2 = 2ag\,(Ce^{2\mu\phi} - \sec^2\phi),$$

C being determined by the initial circumstances. If we now substitute for v^2 in (2), we find that $R \propto e^{2\mu\phi} - \sec^3\phi$. Hence &c.

5008. (By R. Tucker, M.A.)—Parallel chords (PQ, P′Q′, &c.) of a parabola are drawn; and (P, Q) (P′, Q′), &c., are the foci of a series of ellipses which all pass through the focus of the parabola: show that the auxiliary circles all touch the directrix where the circle on the parallel focal chord as diameter touches the directrix.

I. *Solution by* S. A. Renshaw.



Take one of the parallel chords, viz., PQ, and draw PD, Qd perpendiculars to the directrix; join PF, QF, and draw the diameter, to which PQ is an ordinate, to meet it in K, and the directrix in L. Then if, with P and Q as foci, an ellipse be drawn to pass through F, its major axis = PF + QF = PD + Qd (by the parabola) = 2LK. Therefore LK is the radius of the auxiliary circle, which plainly touches the directrix in L; and since QP is any one of the series of parallel chords, therefore, &c.

In Art. 38 of my work on the *Cone*, &c., I have proved that the radius of the generating circle of any focal chord is equal to half the said chord; hence the focal itself forms the major axis of one of the ellipses.

II. *Solution by the* Proposer.

Let (x_1, y_1) (x_2, y_2) be the points P, Q, $\tan^{-1} m$ the angle which PQ makes with the axis, and b the distance of point of intersection of PQ with the axis from the vertex; then we have

$$x_1 + x_2 = 2\lambda + \frac{8a}{m^2}, \quad x_1 x_2 = h^2,$$

$$y_1 + y_2 = \frac{4a}{m}, \quad y_1 y_2 = -4ab;$$

hence $\qquad$ major axis $= x_1 + x_2 + 2a = 2\left(h + \frac{2a}{m^2}, \ \frac{2a}{m}\right),$

coordinates of centre of ellipse $= \left(h + \frac{2a}{m^2}, \frac{2a}{m}\right)$.

Equation to auxiliary circle is

$$\left[X - \left(h + \frac{2a}{m^2}\right)\right]^2 + \left(Y - \frac{2a}{m}\right)^2 = \left[a + \left(h + \frac{2a}{m^2}\right)\right]^2;$$

i.e., $$X^2 + Y^2 - 2X\left(h + \frac{2a}{m^2}\right) - \frac{4aY}{m} + \frac{4a^2}{m^2} = a^2 + 2a\left(h + \frac{2a}{m^2}\right).$$

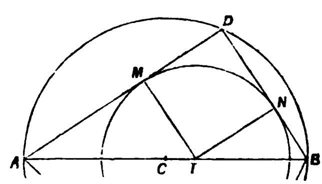
Now write $X = -a$, and the equation becomes

$$\left(Y - \frac{2a}{m}\right)^2 = 0;$$

i.e., the circle touches the directrix at $Y = \frac{2a}{m}$, and this is constant for parallel chords; it is also the point at which the circle on the parallel focal chord as diameter touches the directrix, as it ought to be; for in this case the ellipse becomes the focal chord, and the circle is the auxiliary circle.

5137. (By Prof. SCHEFFER.)—If r and R be the radii of the inscribed and circumscribed circles of a quadrilateral, and h the distance between their centres; prove that $\left(\frac{r}{R+h}\right)^2 + \left(\frac{r}{R-h}\right)^2 = 1$.

I. *Solution by* Prof. EVANS, M.A.; Prof. TANNER, M.A.; J. O'REGAN; R. F. DAVIS, B.A.; *and others.*



Let I, C′ be the centres of the in- and circum-scribed circles; AB their common diameter. Starting at A, draw a quadrilateral circumscribed about I and inscribed in C′. [This is possible, for the quadrilateral of the question is so drawn.] M, N are points in which AD, DB touch I. Then we have

$$\sin MAI = \frac{MI}{AI} = \frac{r}{R+h}, \quad \sin NBI = \frac{NI}{BI} = \frac{r}{R-h}.$$

But MAI, NBI are complementary; therefore

$$\left(\frac{r}{R+h}\right)^2 + \left(\frac{r}{R-h}\right)^2 = 1.$$

[This result is deduced from Jacobi's geometrical construction of the addition theorem in Durège's *Theorie der elliptischen Functionen*, 2nd ed., p. 185, where are also given the corresponding formulæ for a triangle and a pentagon.]

II. *Solution by* R. W. Genese, M.A.

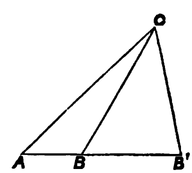
It is well known that, if a quadrilateral can be inscribed in one circle and described about another, an infinite number of such can be drawn. Take one vertex on the line of centres, the opposite will be also. Then, if α and $\frac{1}{2}\pi - \alpha$ be the inclinations of the sides to the line of centres, we have
$$(R \pm h) \cos\alpha = r, \quad (R \mp h) \sin\alpha = r;$$
whence the relation in question.

III. *Solution by* Professor Townsend, M.A., F.R.S.

If S and s be the two circles, O and o their two centres, R and r their two radii, AB and ab their two diameters on their line of centres Oo, and h as above the distance Oo; then, taking for the arbitrary position of the quadrilateral inscribed to S and circumscribed to s, that for which A and B are a pair of opposite vertices, and therefore the angles at the remaining two C and D, determined by the pair of tangents from A and B to S, both right angles, we have at once, by similar triangles,
$$\frac{r}{R \pm h} = \frac{AC}{AB} \text{ and } \frac{r}{R \mp h} = \frac{BC}{AB},$$
and as, by the right angle at C, $AC^2 + BC^2 = AB^2$, therefore &c.

5172. (By R. Tucker, M.A.)—If Δ, Δ' be the areas of the two triangles in the "ambiguous" case (given A, a, b), prove that the continued product of the inscribed and escribed radii to side b is equal to $\Delta\Delta'$.

I. *Solution by* A. W. Cave; Rev. J. L. Kitchin, M.A.; *and others.*



Let ABC, AB′C be the two triangles, and let
$$AB = c_1, \quad AB' = c_2;$$
then c_1, c_2 are the roots of
$$c^2 - 2bc\cos A + b^2 - a^2 = 0;$$
therefore
$$\begin{aligned}
&(a+b+c_1)(a-b+c_1)(a+b+c_2)(a-b+c_2) \\
&= \left\{(a+b)^2 + (a+b)(c_1+c_2) + c_1c_2\right\}\left\{(a-b)^2 + (a-b)(c_1+c_2) + c_1c_2\right\} \\
&= \left\{a^2 + 2ab + b^2 + 2b(a+b)\cos A + b^2 - a^2\right\} \\
&\quad \times \left\{a^2 - 2ab + b^2 + 2b(a-b)\cos A + b^2 - a^2\right\} \\
&= 2b(a+b)(1+\cos A) \,.\, 2b(b-a)(1-\cos A) \\
&= 4b^2(b^2-a^2)\sin^2 A = 4b^2c_1c_2\sin^2 A = 16\Delta\Delta'.
\end{aligned}$$

Hence the continued product of the radii

$$= \frac{16\Delta^2\Delta'^2}{(a+b+c_1)(a+b+c_2)(a-b+c_1)(a-b+c_2)} = \frac{16\Delta^2\Delta'^2}{16\Delta\Delta'} = \Delta\Delta'.$$

II. *Solution by* Rev. H. G. Day, M.A.; Prof. Evans, M.A.; *the* Proposer; *and many others.*

Let r, r' be the inscribed radii and ρ, ρ' the escribed radii; then

$$r = b \sin \tfrac{1}{2}A \sin \tfrac{1}{2}ACB \sec \tfrac{1}{2}ABC, \quad r' = b \sin \tfrac{1}{2}A \sin \tfrac{1}{2}ACB' \operatorname{cosec} \tfrac{1}{2}ABC,$$

$$\rho = b \cos \tfrac{1}{2}A \cos \tfrac{1}{2}ACB \sec \tfrac{1}{2}ABC, \quad \rho' = b \cos \tfrac{1}{2}A \cos \tfrac{1}{2}ACB' \operatorname{cosec} \tfrac{1}{2}ABC;$$

therefore $rr'\rho\rho' = \tfrac{1}{4}B^4 \sin^2 A \sin ACB \sin ACB' \operatorname{cosec}^2 ABC$

$$= \tfrac{1}{4}a^2b^2 \sin ACB \sin ACB' = \Delta\Delta'.$$

5183. (By Prof. Evans, M.A.)—A cube, each of whose edges is n inches in length, is divided by three systems of parallel red lines into cubic inches; find how many *different* routes of $3n$ inches each, by red lines, are there from one corner of the cube to the corner diagonally opposite.

Solution by E. B. Elliott, M.A.

More generally, if, instead of a cube, a parallelepid m inches by n inches by p inches be taken, the shortest routes between two opposite vertices have each $m+n+p$ inch stages.

Now if $m+n+p$ journeys might be made in any order, the whole number of possible ways of performing them would be $\lfloor m+n+p$.

But in this case m of the inch journeys (those in the direction of the side of m inches) must be made in a particular order—from the np plane to a distance 1 from it, from a distance 1 to a distance 2, and so on. Thus the number of ways must be divided by $\lfloor m$. Again, n of the other journeys must in like manner be in a particular order, and so must the remaining p, so that the number must also be divided by $\lfloor n$ and $\lfloor p$. Thus the number of routes required is

$$\frac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}.$$

In the particular case of a cube this becomes $\dfrac{\lfloor 3n}{(\lfloor n)^3}$.

OUR CHRISTMAS π, AND ITS RECIPE; *by* S. M. DRACH, F.R.A.S.

By Drach for π receipt thus given :—
From three millions take eight thousand seven;
Increase remanet five per cent.;
Results to eight decs. represent.
Multiply remanet in view
By factor twelve thousand minus two;
Product at right of last annex,
You get true π to sixteen decs.

$$\begin{array}{r|r|l} 3000000 & & \\ \underline{8007} & 5983986000 & 8007 = \frac{3}{8}(11^4-6^4). \\ 20)\,2991993 & 2991993 & \\ \underline{14959965} & -5983986 & \\ 314159265 & 35897932014 & \end{array}$$

This to invert, as shown below,
To nine and twenty decs. will go,
And make anti Π-étists glow.

$$16\left\{\begin{array}{l} \cdot 02 \\ -00010\ 00070\ 01013\ 08303\ 0013\overline{0,8303}\left(1\ \dfrac{1105}{10^4}\right) \end{array}\right\}$$

$$+\,9\left\{\begin{array}{r} -00001\ 00002\ 00000\ 00000\ 00000,016 \\ +\,0,20000,00048 \end{array}\right\}.$$

INVERSE π IN-VERSE. *By* S. M. DRACH, F.R.A.S.

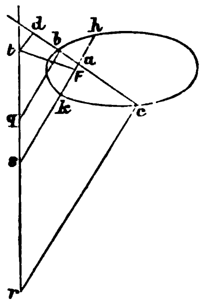
Inverse; (3) 1, (8) 7, (11) 1 subtract
From (1) 2; diff. by sixteen prod. fact.
Also nine times (4) 1, (9) 2 deduct,
Then $\dfrac{1}{\pi}$ to twelve figs. is plucked.
Ben. Gompertz planned these disky fidgits
Denoting zeros before real digits.

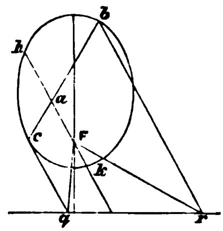
$$\begin{array}{r||r} 02 & +\,32 \\ -00010\ 00070\ 01 & -00160\ 01120\ 16 \\ & \underline{-00009\ 00018\ 00} \\ & 31830\ 98861\ 84 \end{array}$$

5023. (By S. A. RENSHAW.)—Let hFk be any focal chord of a conic of focus F; then, if hk be bisected in a, and any chord be drawn through a, meeting the curve in b and c; and if parallels to h, k be drawn through b and c, meeting the directrix in q and r; prove that Fq, Fr will make equal angles with hk.

Solution by Omakya Chakravarti; E. Rutter; *and many others.*

Let hk and a perpendicular to it through F meet the directrix in s and t respectively; and through t draw td parallel to hk, meeting bc in d.





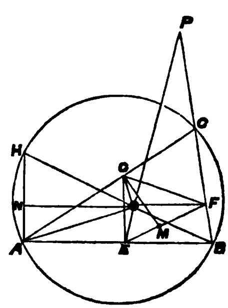
Then, since t is the polar of hk, the polar of a passes through t; and since a is the middle point of hk, the polar of a is parallel to hk; therefore td is the polar of a; therefore $[dbac]$ is harmonic; therefore also $[tqsr]$ is harmonic, and therefore also F $[tqsr]$.

But tFs is a right angle; therefore Fs bisects the angle qFr.

4679. (By H. Murphy.)—Find the locus of a point, such that if perpendiculars be drawn therefrom to the sides of a triangle, the triangle formed by joining the feet of the perpendiculars may be given or a maximum.

Solution by the Proposer.

Let O be a point on the locus; produce EO and BC to meet in P; draw BO, and produce it to meet the circumference in H; and draw the perpendiculars GM and ON. Then, since the triangles EPF, BOP are similar, EF : OB = EP : PB, which is given from the species of EPB; therefore EF : OB is given, and EF . GM is known; therefore GM . OB is given. Again, since the triangles GEM, ANO are similar, GE : AO = GM : ON; hence BO . ON is given. But ON : OH is known, the species of HNO being known; therefore BO . OH becomes known; hence the problem is reduced to finding the locus of a point on a circle such that the rectangle of the segments



of chords passing through it may be given or a maximum, which is well known.

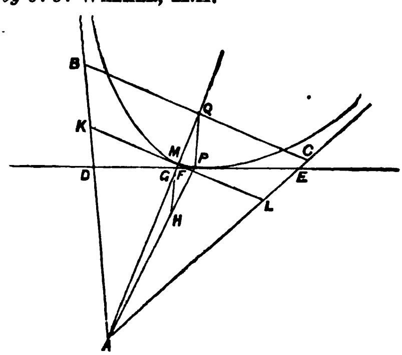
[An extended form of the problem has been given by the EDITOR in Question 1279, solutions of which may be seen on pp. 95—97 of Vol. XVIII. of the *Reprint*.]

5133. (By Prof. TOWNSEND, F.R.S.)—A cone of revolution of uniform density being supposed to float in free equilibrium, with its vertex downwards and its base entirely unsubmerged, in a gravitating fluid of greater density, show that—

(*a*) The sphere having its axis for diameter, intersects its surface in the limiting circle of floatation consistent with the stability of its equilibrium in the erect position of its axis; and determines, at the same time, the limiting density of the fluid consistent with the reality of a second position of equilibrium in which its axis is inclined.

(*b*) For every greater density of the fluid, consistent with the assumed conditions of floatation, there will be an inclined as well as the erect position of equilibrium of the solid, the equilibrium corresponding to the former being stable and to the latter unstable; while for every lesser density, down to that of the solid, there will be but the erect position, for which the equilibrium will be stable.

I. *Solution by* J. J. WALKER, M.A.



Let ABC, ADE be the sections of the whole cone and submerged part by a vertical plane through the axis (AQ) of the former, G and H being their centres of inertia. The plane of flotation, as the axis rotates from the vertical into any inclined position, cutting off a constant volume from the given cone, will touch (at P) a hyperboloid, to which that cone is asymptotic, and DE will consequently touch, at the same point, a hyperbola having AB, AC as asymptotes, and therefore AD . AE $= AK^2$, KL being a perpendicular to the axis through the vertex (M) of the hyperbola.

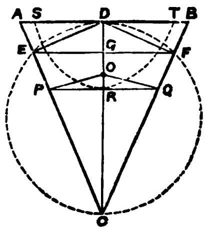
The inclined position of equilibrium for any given density, consistent with such a position, will be determined by HG, and therefore PQ, being perpendicular to DE, and therefore $AQ . AF = AD . AE = AK^2$, F being the point in which the axis AQ meets DE. Hence AQ . AM is greater than AK^2, and the perpendicular from Q on AB would therefore fall between K and B. Thus it appears that the density of the fluid, which

varies inversely as the cube of AM, must be greater than that determined by the sphere described on AQ as diameter, the latter density corresponding to an evanescent inclination of the axis to the vertical in the position of equilibrium.

That the equilibrium in the inclined position of the axis—when possible—is stable, that in the upright position unstable, appears thus:—If the axis becomes less inclined to the vertex, so that D, E, F, H, P become D′, E′, F′, H′, P′ respectively, AF′ being less than AF, and therefore AQ . AF′ less than AD′ . AE′, the perpendicular to D′E′ through P′ must cut AQ below Q, and consequently the vertical through H′ must cut the axis below G, since H′G is parallel to P′Q, whereas, if the axis became more inclined, the segment AF′ becoming less than AF, the metacentre would fall above G.

Lastly, if AQ . AM were less than AK^2, and the density of the fluid therefore less than the limit specified above, in any inclined position of the axis, AQ . AF would be less than AD . AE, the perpendicular to DE at P would therefore meet the axis above Q, and the metacentre fall above G. The unique position of equilibrium, in which the axis is upright, is therefore a stable one.

II. *Solution by the* PROPOSER.



If CD be the axis of the cone, supposed to be in its erect position of equilibrium in the fluid, O its centre of inertia, distant from C by the interval CO = $\frac{3}{4}$CD, ACB any section of it by a vertical plane through CD, EF the section by the same of the plane of floatation corresponding to the position, R the centre of inertia of the displaced volume of the fluid distant from C by the interval CR = $\frac{3}{4}$CG, and SRT the section by the vertical plane of the hyperboloid of revolution, asymptotic to the cone, which is the locus of the different positions of R for all positions of the solid displacing the same volume of the fluid; then, as the elevation RO of the centre of inertia O of the solid above that R of the displaced fluid must for stability of equilibrium in the erect position be less, and for the reality of an inclined as well as of the erect position of equilibrium be greater than the radius of curvature $RP^2 \div RC$ of the hyperboloid at its vertex R, therefore, for the limiting position as regards both, RO must be equal to $RP^2 \div RC$; consequently the two angles at P and Q of the quadrilateral CPOQ must be right angles, therefore so must also the two at E and F of the quadrilateral CEDF, and therefore &c.

The limiting density of the fluid, above which the base of the cone in its oblique position of equilibrium would be partially submerged, may be determined readily as follows. Through any point A of the base draw the plane perpendicular to the opposite edge CB of the cone; the plane so drawn would manifestly be a possible plane of floatation for which the body would float in equilibrium in a fluid of the proper density, having its base in contact with the surface at A and its opposite edge CB vertical, and the corresponding fluid density would consequently be that required.

N.B.—In precisely the same manner it may be shown that the circular cylinder having for diametral the axial plane of an isosceles prism of uniform density, floating in free equilibrium, with its vertical edge downwards and its basal face entirely unsubmerged, in a gravitating fluid of greater density, intersects the equal faces of the prism in the limiting

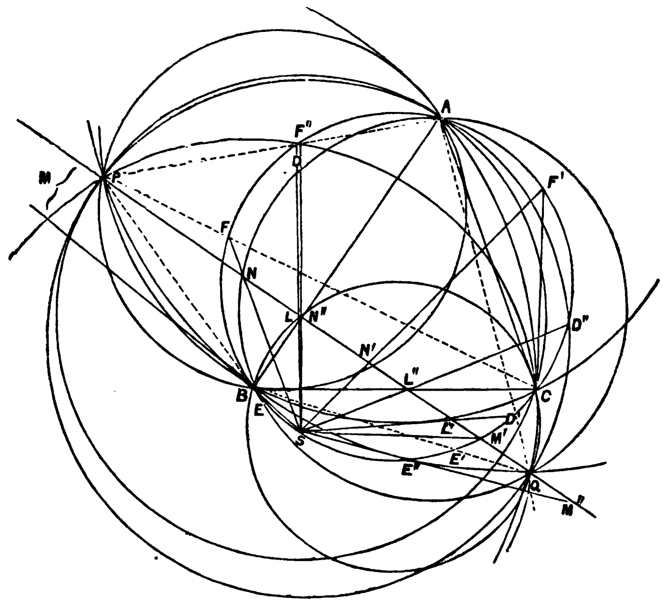
plane of floatation consistent with the stability of the equilibrium in its erect position; and determines, at the same time, the limiting density of the fluid consistent with the reality of an inclined as well as of the erect position of equilibrium of the body.

5048. (By J. L. McKenzie.)—1. ABC is a given triangle, P and Q two given points. The lines AP, BP, CP cut the circumscribing circle ABC in D, E, and F respectively; and the circles BQC, CQA, AQB cut the line PQ in L, M, and N. Prove that the lines DL, EM, FN meet in a point S on the circle ABC.

2. If the lines AQ, BQ, CQ cut the circle ABC in D′, E′, F′; and the circles BPC, CPA, and APB cut PQ in L′, M′, N′; then the lines D′L′, E′M′, F′N′ will intersect in the same point S.

3. If the circles APQ, BPQ, CPQ, cut the circle ABC again in D″, E″, F″; and the lines BC, CA, and AB cut PQ in L″, M″, N″; the lines D″L″, E″M″, F″N″ will also meet in S.

I. *Solution by the* Proposer.



1. It has been proved by Prof. Sylvester (see Salmon's *Higher Plane Curves*, 2nd Ed., Art. 160) that if, starting with a given system P of

$3p+1$ points on a cubic, we describe through them any arbitrary curve, of the order $p+q$, meeting the cubic again in the $3q-1$ points Q; and through the system Q we draw any curve of the order $3q+r$, meeting the cubic again in the $3r+1$ points R, and so on; after any even number of stages, we may reach a residual system consisting of a single point S, which will always be the same, whatever may be the particular method by which it has been reached.

2. Now, the circle ABC and the line PQ form a cubic U passing through the seven points I, J (the circular points), A, B, C, P, and Q. If we draw a circle through any three of the five points A, B, C, P, Q, and a line through the remaining two, we obtain another cubic which must meet U in two points D and L. The line DL meets U in a single residual point S, which is therefore fixed. Or, any circular cubic may be drawn through ABCPQ, meeting U in D and L, then DL will pass through S.

3. Again, the conic through the points ABCPQ and the line at infinity form a cubic, meeting the circle ABC again in a point H, and the line PQ at infinity; therefore the line through H parallel to PQ will pass through S.

II. *Solution by* S. A. Renshaw.

1. Complete the figure by joining MA, LB, QC; and let QC cut the

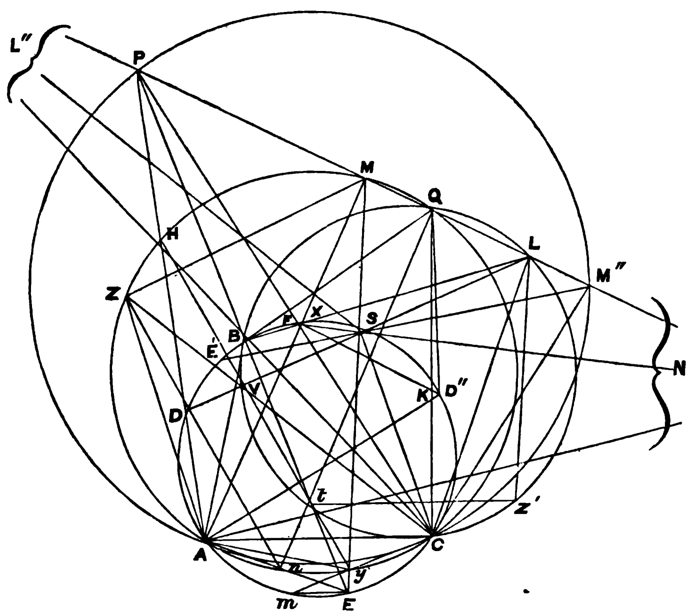


Fig 1.

circle ABC again in K; then, if K be joined with either the intersection of MA or LB with the circle ABC, it is evident that the joining line will

in both cases be parallel to PQ; and therefore X, the intersection of MA and LB, lies on the circle ABC (see Addendum to Potts' *Euclid*, Lem. I., p. 95). We may now easily prove that the point S lies on the circle ABC; for, supposing S to be, first, the intersection of LD and ME, then, since P, M, L are in a straight line, the anharmonic ratio P (ABXM) = P (ABXL). But P (ABXM) = E (ABXS), and P (ABXL) = D (ABXL) = D (ABXS); therefore E (ABXS) = D . ABXS; hence the points D, E, A, B, X, and therefore also S, are on the circle.

In precisely the same way it may be shown that the intersection of FN and LD lies on the circle and therefore coincides with S. (See Mulcahy's *Modern Geometry*, p. 21; and *Reprint*, Vol. XXIII., p. 95.)

2. Again, having drawn DL intersecting the circle ABC in S, join E′S, SM′; then E′SM′ is a straight line. For, since

$$\angle PM'C = 180° - PAC = 180° - CSL,$$

the points C, L, S, M are in a circle.

Therefore $\quad \angle M'SC = M'LC = 180° - E'BC = 180° - E'SC,$

that is, $M'SC + E'SC = 180°$, therefore E′, S, M′ are in one straight line.

Otherwise: Since the $\angle PM'A = FCA = FSA$, therefore $\angle AM'N = ASN$, therefore A, S, M′, N are in a circle, therefore $ASM' + ANM = 180°$. But $ANM' = E'BA = ESA$, therefore $E'SA + ASM' = 180°$, therefore, as before, E′, S, M are in a straight line. Hence E′M′ passes through S, the intersection of DL or FN with the circle, and, by (1), D′L′, E′M′, F′N′ all intersect in the same point on the circle, therefore they all pass through the same point S found in (1).

3. Lastly, produce BC to meet PQ in L″, and join L″S meeting the circle ABC in D″. Then, since D″L″. L″S = CL″. L″B = LL″. L″Q, the points S, D″, Q, L are in a circle, and the $\angle QD''S = QLS = SCM$, since S, L, M′, C are in a circle [see (2)], and $\angle SD''A = SCA$, therefore $\angle QD''A = ACM'$. But A, C, M′, P are in a circle, therefore so are A, D″, Q, P; that is, D″ being the intersection of the circles ABC, APQ, D″L″, passes through S. The other cases, being perfectly symmetrical, may be, *mutatis mutandis*, proved in the same manner.

4. Connected with the further completion of the figure are the following properties:—

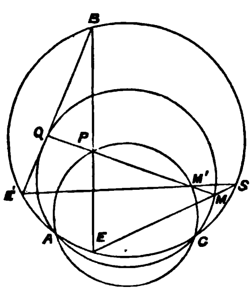


Fig 2.

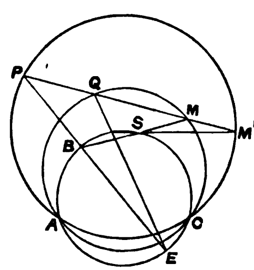


Fig 3.

(*a*). If LD and ME meet the circles BQC and AQC in V and Y re-

spectively, and CV, CY be joined and produced to meet the circles AQC, BQC in Z and Z′ respectively, if ZM and Z′L be joined, ZM is parallel to DL and LZ′ to ME. Again, if AE be joined, meeting the circle ACQ in n, Qn joined will pass through t, the intersection of PE or VY with the circle BCQ (Figs. 2 and 3), and the points Y, t, V, H (H being the intersection of AP with the circle ACQ) will be in one right line parallel to Zn. For V, S, C, Y are plainly in a circle, and

$$tQC = tZ'C = tBC = EBC = nAC = nQC,$$

therefore n, t, Q are in a right line; and since

$$\angle tVC = YSC = ESC = EAC = nZC,$$

therefore YtV is parallel to Zn; but since the $\angle$ DSE = 180° − DAE, and also = ZMY (since ZM is parallel to DS) = 180° − ZAY, therefore $\angle$ HAn = ZAY, and the segment ZH = nY; consequently Zn is parallel to HY, therefore HY coincides with VtY, and H, V, t, Y are in the same straight line. It is easy to see also that tZ′, nY, and mE are parallel.

(β). From the proof that has been given of the second part of the theorem, the following may be deduced; viz., "If three circles pass through the same two points (A and C of Fig. 3), and a chord PM′ of one of the circles be drawn, and meeting another of the circles in M and Q; also, if from P any secant be drawn cutting the third circle in B and E, and QB joined meets this third circle in E′; then E′M′, EM intersect in a point S′ upon the third circle."

5158. (By Prof. Tanner, M.A.)—Find the points on an ellipsoid such that the projection of the indicatrix on the plane xy may be a circle.

Solution by G. S. Carr; E. B. Seitz; *and others.*

Let a, b, c be the semi-axes of the ellipsoid, a being the greatest.

A parallel central section is an ellipse similar to the indicatrix. The projection of this ellipse on the plane of xy is to be a circle. Therefore, if the centre be the origin, the minor axis of the ellipse must lie in the plane of xy. If also the axes of coordinates coincide with a, b, c, this can only happen when the cutting plane passes through the y axis. Thus, b is the minor axis of the ellipse, and the projection of the major upon a must be equal to b. Therefore the coordinates of the point on the surface conjugate to the elliptic section are

$$z = \frac{bc}{a}, \quad x = (a^2 - b^2c^2)^{\frac{1}{2}}, \quad y = 0.$$

There are three other points similarly situated in the plane of xz.

If the ellipsoid be referred to other rectangular axes through its centre, then, taking for its equation

$$\phi = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the directions of the principal diameters of the central section parallel to the indicatrix are given by

$$\begin{vmatrix} \phi_x & \phi_y & \phi_z \\ x & y & z \\ l & m & n \end{vmatrix} = 0,$$

where $lx+my+nz=0$ is the cutting plane; three equations, including that of the surface for determining (x, y, z) the extremity of a principal diameter of the section.

Since one of these diameters is to lie in the plane xy, we may put $z=0$. Thus $lx+my+nz=0$, and we may write $x=m$, $y=-l$, $z=0$ in the expanded determinant in order to obtain l, m, n, which will define the situation of the required point on the surface.

5134. (By Prof. WOLSTENHOLME, M.A.)—Prove the following construction for finding a point P in the base BC of a triangle ABC such that the ratio $AP^2 : BP.PC$ has a given value:—If O be the centre of the circle ABC, divide AO in L such that AL : LO has the given value; then the circle whose centre is L and radius LA will meet BC in two points satisfying the condition. The given value must be not less than $(b+c)^2-a^2 : a^2$, for which value the circle touches BC at the point where the bisector of the angle A meets it; and for any greater value the circle meets BC in two points P, Q such that AP, AQ are equally inclined to the bisector.

I. *Solution by* Professor TOWNSEND, F.R.S.

The several circles in the above construction, for different values of the ratio, all touching at the vertex A that circumscribing the triangle ABC, and consequently forming with it a coaxal system whose limiting points coincide at A, we have therefore [see Townsend's *Modern Geometry*, Art. 192], for any one of them in general having its centre at L and intersecting BC at two real and distinct points P and Q,

$$PB.PC : PA^2 = QB.QC : QA^2 = LO : LA,$$

and for the two of them in particular having their centres at M and N and touching BC internally and externally at R and S respectively,

$$BR^2 : BA^2 = CR^2 : CA^2 = OM : OA$$

and $$BS^2 : BA^2 = CS^2 : CA^2 = ON : OA;$$

which latter determining obviously the two limiting values of the ratio LA : LO consistent with the reality of the two points of section P and Q, and giving for them $[(b+c)^2-a^2] : a^2$ and $[a^2-(b-c)^2] : a^2$ respectively; therefore, &c.

II. *Solution by* C. Leudesdorf, M.A.; R. E. Riley, B.A.; *and others.*

Let $AP^2 : BP \,.\, PC = \lambda$ and $BP = \mu \,.\, BC$; then, if unit vectors along AB, AC be α, β respectively, we have

$$\lambda = \frac{[c\alpha + \mu(b\beta - c\alpha)]^2}{-\mu(1-\mu)a^2} = \frac{\mu^2a^2 + \mu(b^2-c^2-a^2)+c^2}{-\mu(1-\mu)a^2} \;\ldots\ldots\ldots\ldots (1),$$

$$\left(\text{since } S\alpha\beta = -\cos A = \frac{a^2-b^2-c^2}{2bc}\right).$$

Now, if AO be $m\alpha + n\beta$, $AL = \lambda' \,.\, AO$, the equation to the circle, centre L and radius LA, is

$$[\rho - \lambda'(m\alpha+n\beta)]^2 = \lambda'^2(m\alpha+n\beta)^2, \;\text{ or }\; \rho^2 = 2\lambda' S(m\alpha+n\beta)\rho,$$

and, for the points where it meets BC, we have

$$\rho = c\alpha + \mu(b\beta - c\alpha);$$

therefore $\mu^2a^2 + \mu(b^2-c^2-a^2) + c^2 = 2\lambda' S(m\alpha+n\beta)[c(1-\mu)\alpha + \mu b\beta]$

$$= 2\lambda'[\mu a(m\cos B - n\cos C) - C(m + n\cos A)] \;\ldots\ldots\ldots (2).$$

But it is easily shown that, since $OA = OB = OC$, $\dfrac{m}{\cos B} = \dfrac{n}{\cos C} = \dfrac{a}{2\sin^2 A}$; and substituting these values in (2), we find $-\lambda'[c^2 + \mu(b^2-c^2)]$, so that

$$\mu^2a^2 + \mu[(b^2-c^2)(1-\lambda') - a^2] + c^2(1-\lambda') = 0 \;\ldots\ldots\ldots\ldots\ldots (3).$$

Now, multiplying out equation (1) and comparing it with (3), we find that they are identical if $1+\lambda = \dfrac{1}{1-\lambda'}$; and then

$$\frac{AL}{LO} = \frac{\lambda'}{1-\lambda'} = \lambda = \frac{AP^2}{BP \,.\, CP},$$

which proves the construction.

Now, since the roots of (1) and (3) must be real,

$$[b^2 - c^2 - a^2(1+\lambda)] > 4a^2c^2(1+\lambda)$$

or

$$\left[1+\lambda - \left(\frac{b-c}{a}\right)^2\right]\left[1+\lambda - \left(\frac{b+c}{a}\right)^2\right] > 0,$$

so that $1+\lambda$ must not be $< \left(\dfrac{b+c}{a}\right)^2$.

Again, if AP, AQ be called ρ_1, ρ_2,

$$U\rho_1 + U\rho_2 = \frac{c(1-\mu_1) + \mu_1 b\beta}{[\mu_1^2a^2 + \mu_1(b^2-c^2-a^2)+c^2]^{\frac{1}{2}}} + \text{ a similar quantity in } \mu_2,$$

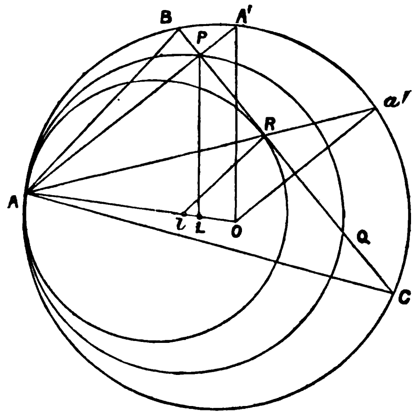
μ_1, μ_2 being the roots of (1),

$$= \frac{1}{a\lambda^{\frac{1}{2}}}\left\{[\mu_1(1-\mu_2)]^{\frac{1}{2}} + [\mu_2(1-\mu_1)]^{\frac{1}{2}}\right\}\left\{\frac{c\alpha}{\mu_1\mu_2} + \frac{b\beta}{(1-\mu_1)^{\frac{1}{2}}(1-\mu_2)^{\frac{1}{2}}}\right\}$$

$$= \frac{(1+\lambda)^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\left\{[\mu_1(1-\mu_2)]^{\frac{1}{2}} + [\mu_2(1-\mu_1)]\right\}(\alpha+\beta).$$

Thus $U\rho_1 + U\rho_2$ is a multiple of $\alpha+\beta$, therefore AP, AQ are equally inclined to the bisector of the angle BAC; when the two values of μ are equal, then AP, AQ coincide with each other and with the bisector.

III. *Solution by the* Proposer.



Draw the circle ABC, and let O be its centre, L the centre of a circle touching the former at A, and meeting BC in P, Q; join AP and produce it to meet the circle ABC in A′, and join LP, OA′. Then $AP^2 : BP \, . \, PC = AP^2 : AP \, . \, PA' = AP : PA' = AL : LO$, since OA′, LP will be parallel. This proves the construction. Of course $AQ^2 : BQ \, . \, QC$ in an equal ratio, so that generally two points can be found in the base satisfying the conditions. The limiting case will be when the second circle touches BC (in R suppose); let l, a' be the corresponding positions of L, A′, then lR, Oa' will be perpendicular to BC, or a' bisects the arc BC, and therefore AR bisects the angle A; hence it will also bisect the angle PAQ, or AP, AQ are equally inclined to the bisector of A. The minimum value of the ratio is then

$$AR^2 : BR : RQ = \sin B \sin C : \sin^2 \tfrac{1}{2}A$$
$$= 2 \sin B \sin C : 1 - \cos A = 2 \sin B \sin C \, (1 + \cos A) : \sin^2 A$$
$$= 2bc \, (1 + \cos A) : a^2 = (b+c)^2 - a^2 : a^2.$$

The construction holds whether L be in AO or in AO produced either through O or A; but in the latter case we shall again have a limit where the circle touches BC, corresponding to the ratio

$$\sin B \sin C : -\cos^2 \tfrac{1}{2}A \quad \text{or} \quad (b-c)^2 - a^2 : a^2.$$

If the proposed ratio be negative and numerically greater than 1, the centre lies in AO produced through O, and the points are always real. If negative and numerically less than 1, the centre will lie on OA produced through A, and the points will only be real if the ratio be numerically greater than $a^2 - (b-c)^2 : a^2$; and will both lie on CB produced through B, if $B > C$.

If the tangent at A to the circle ABC meet BC in D, and it be required to find a point P in the straight line BC such that the ratio $AP^2 : BP \, . \, PC$ has given value m, then, if m be positive and $< (b+c)^2 - a^2 : a^2$, the problem is impossible.

If $m > \dfrac{(b+c)^2 - a^2}{a^2}$, the points are real and lie both in BC.

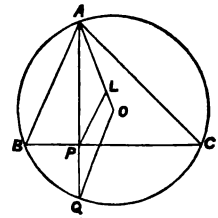
If $0 > m > \dfrac{(b-c)^2 - a^2}{a^2}$, the problem is impossible.

If $-1 < m < \dfrac{(b-c)^2 - a^2}{a^2}$, the points are real and both lie in the part beyond D.

If $m < -1$, the points are real and one lies in BD, the other in the part beyond C (C ∞).

IV. *Solution by* R. F. DAVIS, B.A.; E. RUTTER; *and others.*

Produce AP to meet the circumscribing circle in Q. Then we have



$$AP : PQ = AP^2 : AP . PQ$$
$$= AP^2 : BP . PC :: AL : LO.$$

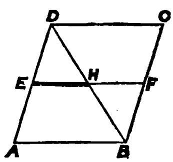
Thus LP is parallel to OQ; and since OA = OQ, LA = LP. If AP′Q′ be the other position of APQ corresponding to the fixed ratio, we have

$$AP : PQ = AP' : P'Q';$$

therefore QQ′ is parallel to BC, and Q, Q′ are equidistant from the middle point of the arc BC, &c. &c.

5155. (By Professor CROFTON, F.R.S.)—If the base AB of a parallelogram or trapezoid ABCD is horizontal, and two points X, Y are taken at random, in the triangles BDA, BDC respectively, show that the chance that X is higher than Y is $\frac{1}{6}$.

I. *Solution by* HUGH MCCOLL, B.A.; Rev. H. L. DAY, M.A.; *and others.*

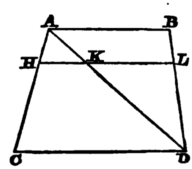


Let the height of the point D from the base AB be the linear unit; then, dividing the triangle BDA into infinitesimal strips (of thickness dx) parallel to the base, and denoting the height of any such strip (as EH) by x, the chance that the point X falls in this strip is $2(1-x)\,dx$. Given that X falls in the strip EH, the chance that Y falls in the triangle HBF is

$$\Delta HBF \div \Delta DBC = x^2.$$

The required chance is therefore $\displaystyle\int_0^1 2(1-x)\,x^2\,dx = \tfrac{1}{6}.$

II. *Solution by* Colonel A. R. CLARKE, C.B., F.R.S.



Let Δ, Δ' be the areas of the triangles ACD, ABD, p the distance between the sides AB, CD, to each of which HKL is parallel at the distance x from A. Suppose the triangle ACD divided into strips of infinitesimal breadths ι, then, if X fall in the strip ιHK, Y must fall in KLD ($= \nabla$), and the total number of favourable cases is the sum of the quantities

$$HK . \iota\nabla = \frac{x}{p}\,CD . \iota\nabla = \frac{2\Delta}{p^2} . x\iota\nabla.$$

Imagine the triangle ∇ to rotate round KL until it is perpendicular to ABCD, then, as x varies from 0 to p, this triangle will generate a tetra-

hedron, standing on the base ABD, and having a height $= p$. Then clearly the sum of the products $xi\nabla$ is the volume $(\frac{1}{3}p\Delta')$ of the solid multiplied by the distance $(\frac{1}{4}p)$ of its centroid from a plane through AB perpendicular to ABCD. Hence the number of favourable cases is $2\dfrac{\Delta}{p^2}\cdot\frac{1}{12}p^2\Delta'$, and the total number of cases being $\Delta\Delta'$, the chance is $\frac{1}{6}$.

5123. (By J. L. McKenzie.)—If Δ be the area of a triangle, and l, m, n the lines drawn from the angles to the middle points of the opposite sides, prove that

$$\begin{aligned}18\Delta = {} & a\sin C\,(4l^2 - a^2\sin^2 C)^{\frac{1}{2}} + a\sin B\,(4l^2 - a^2\sin^2 B)^{\frac{1}{2}} \\ & + b\sin A\,(4m^2 - b^2\sin^2 A)^{\frac{1}{2}} + b\sin C\,(4m^2 - b^2\sin^2 C)^{\frac{1}{2}} \\ & + c\sin B\,(4n^2 - c^2\sin^2 B)^{\frac{1}{2}} + c\sin A\,(4n^2 - c^2\sin^2 A)^{\frac{1}{2}};\end{aligned}$$

$$\begin{aligned}24\Delta = {} & 2c\sin A\,(4l^2 - a^2\sin^2 C)^{\frac{1}{2}} + 2a\sin B\,(4m^2 - b^2\sin^2 A)^{\frac{1}{2}} \\ & + 2b\sin C\,(4n^2 - c^2\sin^2 B)^{\frac{1}{2}} + a^2\sin 2C + b^2\sin 2A + c^2\sin 2B.\end{aligned}$$

Solution by Elizabeth Blackwood; A. W. Cave, B.A.; Professor Evans, M.A.; *and others.*

Since the square of the distance between the points $(\alpha_1\beta_1\gamma_1)$ and $(\alpha_2\beta_2\gamma_2)$

$$= -\frac{abc}{4\Delta^2}\left\{a(\beta_1-\beta_2)(\gamma_1-\gamma_2) + b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2) + c(\alpha_1-\alpha_2)(\beta_1-\beta_2)\right\},$$

therefore $$l^2 = -\frac{abc}{4\Delta^2}\left(a\frac{\Delta^2}{bc} - b\frac{2\Delta^2}{ca} - c\frac{2\Delta^2}{ab}\right);$$

therefore $$4l^2 - a^2\sin^2 C = 2b^2 + 2c^2 - a^2 - a^2\sin^2 C = \left(\frac{3b^2+c^2-a^2}{2b}\right)^2.$$

Similarly, $$4l^2 - a^2\sin^2 B = 2b^2 + 2c^2 - a^2 - a^2\sin^2 B = \left(\frac{b^2+3c^2-a^2}{2c}\right)^2;$$

therefore $$\begin{aligned}& a\sin C\,(4l^2 - a^2\sin^2 C)^{\frac{1}{2}} + a\sin B\,(4l^2 - a^2\sin^2 B)^{\frac{1}{2}} \\ & = a\sin C\,.\,\frac{3b^2+c^2-a^2}{2b} + a\sin B\,.\,\frac{b^2+3c^2-a^2}{2c} \\ & = \Delta\left(3 + \frac{c^2-a^2}{b^2} + 3 + \frac{b^2-a^2}{c^2}\right);\end{aligned}$$

therefore $$\begin{aligned}& \Delta\left(6 + \frac{c^2-a^2}{b^2} + \frac{b^2-a^2}{c^2}\right) \\ & = a\sin C\,(4l^2 - a^2\sin^2 C)^{\frac{1}{2}} + a\sin B\,(4l^2 - a^2\sin^2 B)^{\frac{1}{2}};\end{aligned}$$

similarly, $$\begin{aligned}& \Delta\left(6 + \frac{a^2-b^2}{c^2} + \frac{c^2-b^2}{a^2}\right) \\ & = b\sin A\,(4m^2 - b^2\sin^2 A)^{\frac{1}{2}} + b\sin C\,(4m^2 - b^2\sin^2 C)^{\frac{1}{2}}, \\ & \Delta\left(6 + \frac{b^2-c^2}{a^2} + \frac{a^2-c^2}{b^2}\right) \\ & = c\sin B\,(4n^2 - c^2\sin^2 B)^{\frac{1}{2}} + c\sin A\,(4n^2 - c^2\sin^2 A)^{\frac{1}{2}}.\end{aligned}$$

Adding these last three results, we have the required result.

Again, $\quad 2c \sin A\, (4l^2 - a^2 \sin^2 C)^{\frac{1}{2}} = 2c \sin A \,.\, \dfrac{3b^2 + c^2 - a^2}{2b}$

$$= 2\Delta \,.\, \frac{3b^2 + c^2 - a^2}{b^2} = 8\Delta - 4\Delta\left(1 - \frac{c \cos A}{b}\right)$$

$$= 8\Delta - 2ab \sin C \,.\, \frac{a \cos C}{b} = 8\Delta - a^2 \sin 2C\,;$$

therefore $\quad 8\Delta = 2c \sin A\,(4l^2 - a^2 \sin^2 C)^{\frac{1}{2}} + a^2 \sin 2C\,;$

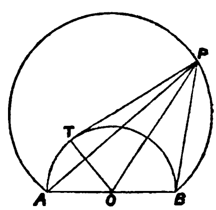
similarly, $\quad 8\Delta = 2a \sin B\,(4m^2 - b^2 \sin^2 A)^{\frac{1}{2}} + b^2 \sin 2A,$

$$8\Delta = 2b \sin C\,(4n^2 - c^2 \sin^2 B)^{\frac{1}{2}} + c^2 \sin 2B.$$

Adding these last three results, we have the required result.

5162. (By R. W. Genese, M.A.)—If a semicircle be drawn on AB; prove that the locus of a point P such that the square on the tangent from P to the semicircle varies as PA . PB is a segment of a circle.

Solution by Christine Ladd; R. E. Riley, B.A.; R. Tucker, M.A.; Rev. H. G. Day, M.A.; *and many others.*



Let PT be the tangent at T, and put $\angle APB = \theta$; then, if

$$PA \,.\, PB = \mu \,.\, PT^2,$$

we have

$$2\mu \,.\, PT^2 \cos\theta = PA^2 + PB^2 - AB^2$$
$$= 2\,(OP^2 - OA^2) = 2PT^2\,;$$

hence $\quad \sec\theta = \mu = \text{a constant};$

and therefore the locus of P is a circular arc on AB as base.

5206. (By J. B. Sanders.)—A given weight P draws another given weight W up an inclined plane of known height and length, by means of a string parallel to the plane; find when and where P must cease to act, in order that W may just reach the top of the plane.

Solution by R. Battle, M.A.; R. E. Riley, B.A.; *and others.*

Let h = height of plane, l = length of plane, x = space through which W has moved when P discontinues to act, t = time during which W has

moved when P discontinues to act, v = velocity acquired by W when P discontinues to act, f = accelerating force on the system; then we have

$$f = \frac{\left(P - W\frac{h}{l}\right) g}{P+W} = \frac{(Pl - Wh)\,g}{(P+W)\,l}.$$

As the velocity v is exactly sufficient to take W to the top of the plane, it is equal to the velocity acquired by a body moving through the space $l-x$ down the plane, the accelerating force being $f' = g\,\frac{h}{l}$; therefore

$$2fx = v^2 = 2f'(l-x);$$

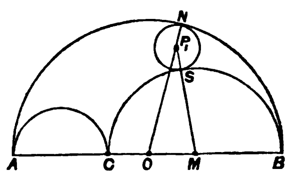
therefore

$$x = \frac{f'l}{f+f'} = \frac{P+W}{P}\left(\frac{hl}{h+l}\right),$$

$$t = \left(\frac{2x}{f}\right)^{\frac{1}{2}} = \frac{(P+W)\,l}{P}\left\{\frac{2Ph}{(Pl-Wh)(h+l)\,g}\right\}^{\frac{1}{2}}.$$

5197. (By W. Godward.)—A point C is taken in the diameter AB of a semicircle, and on AC, BC other semicircles are described; also a series of circles whose centres are P_1, P_2, P_3, ... are constructed such that each one has contact with the preceding and the semicircles on AB, AC; the first circle P_1 having contact with the semicircle on BC. Prove that the loci of P_1, P_2, P_3, ... are ellipses having a common focus.

Solution by Christine Ladd; R. E. Riley, B.A.; *and others.*



Let O be the centre of the fixed semicircle; P_1 the centre of one of the circles referred to. Join OP_1, passing through the point of contact N; and join MP_1, where M is the centre of the semicircle on BC; then

$$OP_1 + MP_1 = ON - P_1N + SM + P_1S$$
$$= ON + SM = \text{constant};$$

therefore the locus of P_1 is an ellipse with foci O, M; and, changing the point C, it is evident the loci will all have the common focus O.

5182. (By Professor Williamson, M.A.)—If a plane vertical area, immersed in a fluid, revolve in its own plane around its centre of gravity, prove that the locus of its centre of pressure is a circle.

I. *Solution by* J. J. Walker, M.A.

This theorem holds when the plane of the area is inclined at any angle, say ϕ, to the horizon.

Take as axes of coordinates the horizontal line and a perpendicular in the plane, through its centre of gravity, h being the distance of that point from the surface measured along the latter line; and let (x, y) be the coordinates of any element (a) of the whole area (A). Now suppose the area to turn about the origin in its own plane through an angle θ, and (x', y') to be the coordinates of the new position of a, (ξ, η) to be those of the centre of pressure;

$$Ah\xi \sin\phi = \Sigma ax'(h+y')\sin\phi,\quad Ah\eta\sin\phi = \Sigma ay'(h+y')\sin\phi.$$

But $\qquad x' = x\cos\theta + y\sin\theta,\quad y' = -x\sin\theta + y\cos\theta$;

whence $\qquad Ah\xi = \Sigma a\,(x\cos\theta + y\sin\theta)\,(h - x\sin\theta + y\cos\theta)$

$$= h\cos\theta\,\Sigma ax + h\sin\theta\,\Sigma ay + \cos 2\theta\,\Sigma axy - \tfrac{1}{2}\sin 2\theta\,\Sigma a\,(x^2-y^2)\ \ldots(1),$$

in which the coefficients of $\cos\theta$, $\sin\theta$ vanish, the origin being the centre of gravity of A. Similarly,

$$Ah\eta = \sin^2\theta\,\Sigma ax^2 + \cos^2\theta\,\Sigma ay^2 - \sin 2\theta\,\Sigma axy\ \ldots\ldots\ldots\ldots\ldots(2).$$

Putting $\qquad \Sigma ax^2 = 2Aha,\quad \Sigma ay^2 = 2Ahb,\quad \Sigma axy = Ahc,$

the result of eliminating θ from (1), (2) is

$$\xi^2 + \{\eta - (a+b)\}^2 = (a-b)^2 + c^2.$$

II. *Solution by* Professor WOLSTENHOLME, M.A.

Let h be the depth of the centre of inertia O below the surface of the fluid, and let (x, y) be vertical and horizontal coordinates of an element $d\Omega$ of the area, measured from O; (ξ, η) coordinates of the same element referred to the principal axes at O. Then, if θ be the angle between Ox, $O\xi$, and the area be taken as unity, we have

$$\bar{x} = \frac{\Sigma(\xi\cos\theta - \eta\sin\theta)^2 . d\Omega}{h},\quad \bar{y} = \frac{\Sigma(\xi\cos\theta - \eta\sin\theta)(\xi\sin\theta + \eta\cos\theta)}{h}.$$

But $\qquad \Sigma(\xi^2 d\Omega) = B,\quad \Sigma(\eta^2 d\Omega) = A,\quad \Sigma(\xi\eta d\Omega) = 0$;

whence $\qquad \bar{x} = \dfrac{B\cos^2\theta + A\sin^2\theta}{h} = \dfrac{B+A}{2h} + \dfrac{B-A}{2h}\cos 2\theta,$

and $\qquad \bar{y} = \dfrac{B-A}{h}\sin\theta\cos\theta = \dfrac{B-A}{2h}\sin 2\theta$;

or the centre of pressure traces out in space a circle whose centre is at a depth $\dfrac{B+A}{2h}$ below the centre of inertia and whose radius is $\dfrac{B-A}{2h}$. If L, M be the highest and lowest positions of the centre of pressure, and LP, MP be drawn parallel each to that principal axis which is horizontal when the centre of pressure is at L or M, P will be the centre of pressure. In this form I set this result at an examination in 1875, but I believe that I first got it from an examination paper at St. John's College, Cambridge, many years ago.

If (ξ, η) be the coordinates of the centre of pressure referred to the principal axes, we have

$$\bar{\xi} = x\cos\theta + y\sin\theta = \frac{B}{h}\cos\theta,\quad \bar{\eta} = x\sin\theta - y\cos\theta = \frac{A}{h}\sin\theta,$$

so that the locus of the centre of pressure in the lamina is an ellipse $\dfrac{x^2}{B^2} + \dfrac{y^2}{A^2} = \dfrac{1}{h^2}$. The circle which is its locus in space touches the auxiliary

circles of this ellipse, and will in general cut the ellipse in four points of which the centre of pressure is the lowest. (For all four points to be real, we must have $B > 2A$, if B be the greater of the two.)

The other three points in which the circle meets the ellipse will form a triangle whose sides are equidistant from the centre of inertia, the distance being $\frac{AB}{h(B-A)}$; the normals to the ellipse at these points will meet in the point
$$\frac{A}{B}\cdot\frac{A+B}{h}\cos\theta,\quad \frac{B}{A}\cdot\frac{A+B}{h}\sin\theta,$$
the centre of inertia of the triangle will be the point
$$\frac{B}{3h}\cdot\frac{B+A}{B-A}\cos\theta,\quad \frac{A}{3h}\cdot\frac{A+B}{A-B}\sin\theta;$$
and the centre of perpendiculars the point
$$\frac{A}{h}\cdot\frac{B+A}{B-A}\cos\theta,\quad \frac{B}{h}\cdot\frac{A+B}{A-B}\sin\theta;$$
all referred to the principal axes at O.

It will be found that, when $B > 2A$, the four points of intersection will all be real so long as $\tan^2\theta < \frac{B}{A}\frac{(B-2A)^3}{(2B-A)^3}$. Another point connected with the triangle is the centre of its nine-point circle which is
$$\frac{(B+A)^2}{2(B-A)}\cos\theta,\quad \frac{(B+A)^2}{2(A-B)}\sin\theta,$$
so that it traces out a circle within the lamina and in space, and the nine-point circle always touches two fixed circles, one of which is, of course, the circle which touches the sides of the triangle, the radius of the other being $\frac{A^2+B^2}{2(B-A)h}$.

5189. (By R. W. Genese, M.A.)—If a hexagon, inscribed in one circle, be described about another, and if R, r be the radii, and h the distance between the centres of the circles, prove that
$$\sin\cos^{-1}\frac{r}{R-h}+\cos\sin^{-1}\frac{r}{R+h}=1.$$

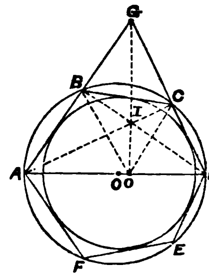
I. *Solution by* Prof. Lloyd Tanner, M.A.

We have [see Durège's *Theorie der elliptischen Functionen*, p. 183]
$$\frac{r}{R+h}=\cos am\,\tfrac{1}{3}K,\quad \frac{r}{R-h}=\frac{\cos am\,\frac{1}{3}K}{\Delta\,am\,\frac{1}{3}K}=\sin am\,\tfrac{2}{3}K;$$
therefore
$$\sin\cos^{-1}\frac{r}{R+h}=\sin am\,\tfrac{1}{3}K=\frac{\cos am\,\frac{2}{3}K}{\Delta\,am\,\frac{2}{3}K},$$
and
$$\cos\sin^{-1}\frac{r}{R-h}=\cos am\,\tfrac{2}{3}K;$$
therefore
$$\sin\cos^{-1}\frac{r}{R+h}+\cos\sin^{-1}\frac{r}{R-h}$$
$$=\frac{\cos am\,\frac{2}{3}K}{\Delta\,am\,\frac{2}{3}K}+\cos am\,\tfrac{2}{3}K=1\ (loc.\ cit.,\ \text{p. }184).$$

This agrees with Mr. Genese's result, for

$$\sin\cos^{-1} u = (1-u^2)^{\frac{1}{2}} = \cos\sin^{-1} u.$$

II. *Solution by* Professor TOWNSEND, M.A., F.R.S.

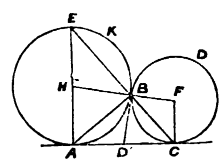


If ABCDEF be the hexagon in the position of symmetry in which its diagonal AD passes through the centres O and o of both circles, since then $Ao = R + h$ and $Do = R - h$, and since the angles ABD and ACD are both right angles, to prove the relation in question, it is only necessary to shew that $AB + CD = AD$, which may be readily done as follows.

Producing AB and CD to meet at G, drawing Bo, Co, and Go, which, of course, bisect the angles ABC, BCD, and AGD respectively, taking on Go the point I at which it is met by the bisectors AI and DI of the angles DAG, and ADG, which is, of course, the centre of the circle inscribed to the triangle AGD, and joining IB and IC; then, since the angles ABo and AIo are equal, each being complementary to IDo, therefore the four points ABIo lie on a circle, and IB = Io: and, since the angles DCo and DIo are equal, each being complementary to IAo, therefore the four points DCIo lie on a circle, and IC = Io; consequently, the four points BGCI lie on a circle, and IB = IC, from which it manifestly follows at once that $AB + CD = AD$, and therefore, &c.

5242. (By R. E. RILEY, B.A.)—A fixed circle touches a fixed straight line at A; any circle is drawn touching the fixed circle at B and the straight line at C; prove that BC passes through a fixed point.

Solution by ARTHUR WINSLOW; E. RUTTER; *and many others.*



Draw any circle BCD touching the fixed circle ABK at the variable point B, and the fixed straight line at the variable point C; then the radii HB, FB of the two circles, drawn to their common point of intersection B, will be in the same straight line. Produce AH and CB till they intersect in some point E, and join FC. Then, in the triangles BCF and BEH,

$\angle EBH = CBF$, and $\angle EHB = BFC$ (because EA is parallel to FC), therefore $\angle BEH = BCF = CBF = EBH$; therefore $HE = HB$.

But HB is a radius of the circle ABK; therefore HE is a radius of the same circle; wherefore E must be the intersection of the diameter through A with the fixed circle, which is a constant point.

[*Otherwise:* Draw the common tangent BD; then, since DA = DB = DC, the angle ABC is a right angle; and therefore the points C, B, and E (the end of the *diameter* AHE) must be in the same straight line.]

On the Musical Scale. *By* Colin Brown.

We take it for granted that the seven notes of the major scale that form an octave—or the *eight* notes, if we count the repetition of the tonic—are at the ordinarily specified intervals of

C (tone), D (tone), E (semitone), F (tone), G (tone), A (tone), B (semitone), C;

and we propose to enquire what light the harmonic series throws upon the question of the precise magnitudes of the said tones and semitones.

Both theory and experiments have proved that, (1) if two strings of the same quality and tension have lengths, in any assigned proportion, say as 3 : 5, then the number of vibrations per second of the former will be to that of the latter as 5 : 3, that is, in the inverse ratio of their lengths; (2) when the ratio of the vibrations of two musical sounds is expressed by small numbers (as 1 : 2, or 2 : 3) the interval is a consonance, which, however, gradually diminishes in degree as the numbers of its ratio of intervals increase (no primes greater than 5 entering into the ratios, ordinarily considered as consonant); (3) every musical sound is accompanied by a series of harmonics, more or less audible, of which the vibration numbers are that of the fundamental sound multiplied by the series of natural numbers 2, 3, 4, 5, 6, 7, 8, 9, ... As these numbers increase, the corresponding sounds, generally speaking, become rapidly fainter, but they have been heard up to the sixteenth, or even, by artificial appliances, a little higher.

For the convenience of some readers, we may remark before proceeding further, that if, for instance, there be three musical sounds, whose vibrations per second are in proportion to the numbers 3 : 4 : 5, then the interval from the first to the second is expressed by the fraction $\frac{4}{3}$, that from the second to the third by $\frac{5}{4}$, while from the first to the third it is $\frac{5}{3}$. Now it is clear that the $\frac{5}{3}$ is the product of the fractions representing the partial intervals, that is $\frac{5}{3} = \frac{4}{3} \cdot \frac{5}{4}$. So also, if we would deduct from a fifth ($\frac{3}{2}$) a minor third ($\frac{6}{5}$), we must divide $\frac{3}{2}$ by $\frac{6}{5}$, the result being $\frac{3}{2} \cdot \frac{5}{6} = \frac{5}{4}$, which is a major third. As $\frac{5}{4}$ means an ascent of a major third, so $\frac{4}{5}$ means a descent of the same amount; a fourth $\frac{4}{3}$, being $= 2 \times \frac{2}{3}$, is equivalent to ascending an octave ($\frac{2}{1}$) and descending a fifth.

The following pairs of intervals are complementary; that is, they together make up an octave:—

Fourth	($\frac{4}{3}$) and fifth	($\frac{3}{2}$),	$\frac{4}{3} \times \frac{3}{2} = 2$;
Major third	($\frac{5}{4}$) and minor sixth	($\frac{8}{5}$),	$\frac{5}{4} \times \frac{8}{5} = 2$;
Minor third	($\frac{6}{5}$) and major sixth	($\frac{5}{3}$),	$\frac{6}{5} \times \frac{5}{3} = 2$.

Now if, for the reason specified above, we abstract from the harmonic series—represented by the series of natural numbers—all sounds which depend on primes greater than 5, the series is limited to

2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16 ...... and so on.

Taking the first three (2, 3, 4), we have an octave (2 : 4) divided into two intervals, 2 : 3 and 3 : 4, which we recognise as a fifth and a fourth. This fixes the note G thus:—

C,	D,	E,	F,	G,	A,	B,	C;
1				$\frac{3}{2}$.			2

the notes in small capitals being still undefined. Next, take the octave 3, 4, 5, 6. Here we have the octave divided into three successive intervals, 3 : 4, 4 : 5, and 5 : 6, which we recognise as a fourth, a major third, and a

minor third. The two new notes introduced by this division are F and A, and their positions in the scale are fixed thus:—

C,	D,	E,	F,	G,	A,	B,	C.
1			$\frac{4}{3}$		$\frac{5}{3}$		2

Next, the octave 4, 5, 6, 8 is divided into a major third, a minor third, and a fourth, introducing one new note at the interval of a major third from C. This is clearly E, and we have the notes fixed thus:—

C,	D,	E,	F,	G,	A,	B,	C;
1		$\frac{5}{4}$		$\frac{3}{2}$			2

shewing that C, E, G are related similarly to F, A, C. Next take the octave 5, 6, 8, 9, 10. The first interval here is a minor third 5 : 6; hence we are in a minor scale, and we leave this for subsequent consideration. Take the octave 6, 8, 9, 10, 12; this confirms F, G, and A, but gives no new note. Take 8, 9, 10, 12, 15, 16; here the 9 and 15 are the only new notes; they are, in fact, D and B, completing the values of the notes in the major scale as below:—

C,	D,	E,	F,	G,	A,	B,	C.
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

This finally determines the relative vibration numbers of the seven notes of the scale. As we now see, all might have been derived from the harmonics 6, 8, 9, 10, 12, 15, 16; but it is very interesting to note the manner in which the octave is *gradually* subdivided: first the dominant G; then F and A in the subdominant triad; then E and G in the tonic triad; and then B and D in the dominant triad G, B, D.

The numbers 4, 5, 6 of the harmonic series represent the major triad (either C, E, G; G, B, D; or F, A, C). The next three (omitting 7) 8, 9, 10 represent the first three tones of the scale. If we reduce to a common denominator the fractions just obtained for the different notes of the scale, we get for the relative vibration numbers the following:—

C,	D,	E,	F,	G,	A,	B,	C
24	27	30	32	36	40	45	48

It is important to remark,—and it is a neat way of establishing the scale,—that if we start with the vibration numbers of the three notes C, E, G, we can obtain those of the other notes by associating with this triad the dominant triad G, B, D, and the sub-dominant triad F, A, C. I mark in small capitals the notes thus inferred:—

C,		E,		G,		B,		D
24	(4 : 5)	30	(5 : 6)	36	(4 : 5)	45	(5 : 6)	54
				F,		A,		C
				32	(4 : 5)	40	(5 : 6)	48

Lowering D by an octave, we get the vibration numbers of B, D, F, A, as before.

Though it is not possible, except under certain circumstances, to hear harmonics above the 16th (and the limit of audibility is, roughly speaking, thereabouts), yet we may carry our harmonic series arithmetically to any extent; thus, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48 .... Here, if we except the one number 25, we remark that (1) the eight numbers 24, 27, 30, 32, 36, 40, 45, 48 correspond to the vibration numbers of the eight notes of the scale as just determined; (2) the octave 20, 24, 27, 30, 32, 36, 40 contains, an octave higher, the octave 5, 6, 8, 9, 10, with two additional notes, 27, 30; and if we supply a second to this scale from the

relation 40, 45, 48, by taking 45 and octave lower, we have the scale of A minor, the relative minor of C—that is, we have the *descending* form of that scale. The number 25 enters into the Voice-Harmonium, and a digital corresponding to it enables one to play the minor scale in its ascending form.

The seven successive intervals of the major scale are therefore the following:—

$$\tfrac{9}{8},\ \tfrac{10}{9},\ \tfrac{16}{15},\ \tfrac{9}{8},\ \tfrac{10}{9},\ \tfrac{9}{8},\ \tfrac{16}{15}.$$

Here we have three different intervals—a large tone, represented by $\frac{9}{8}$; a lesser tone, represented by $\frac{10}{9}$; and a small tone, represented by $\frac{16}{15}$.

Whatever, then, may be the pitch of the tonic or key-note of the major scale, for every 24 vibrations of that sound the second vibrates 27 times and the third 30 times, these three numbers being as 8 : 9 : 10; the fourth, fifth, and sixth vibrate 32, 36, 40 times respectively, which again are as 8 : 9 : 10; whilst the relation of the third to the fourth, or of the seventh (45) to the eighth (48), is 15 : 16.

By adding the simple elementary intervals represented by the ratios 8 : 9, 9 : 10, 15 : 16 (that is, as explained, by *multiplying* the corresponding fractions) every relation and interval used in music can be produced with musical and mathematical exactness. The intervals formed by various musical relations within the octave are eighteen in number, each of the six separate intervals existing in three different forms, which, by putting $a = \frac{9}{8}$, $b = \frac{10}{9}$, $c = \frac{16}{15}$, will be as given in the following Table:—

Intervals in the Octave with their Ratios.	Designation.
$a = \frac{9}{8}$	Large tone
$b = \frac{10}{9}$	Less tone
$c = \frac{16}{15}$	Small tone or diatonic semitone
$ab = \frac{5}{4}$	Third, major
$ac = \frac{6}{5}$	„ minor
$bc = \frac{32}{27}$	„ minor, grave
$abc = \frac{4}{3}$	Fourth, perfect
$aba = \frac{45}{32}$	„ pluperfect
$aca = \frac{27}{20}$	„ acute
$abca = \frac{3}{2}$	Fifth, perfect
$bcab = \frac{40}{27}$	„ grave
$cabc = \frac{64}{45}$	„ imperfect
$abcab = \frac{5}{3}$	Sixth, major
$cabac = \frac{8}{5}$	„ minor
$abaca = \frac{27}{16}$	„ major, acute
$abcaba = \frac{15}{8}$	Seventh, major
$bcabac = \frac{16}{9}$	„ minor
$cabaca = \frac{9}{5}$	„ minor, acute
$abcabac = 2$	Octave

By taking the *differences* of the small intervals; that is, by dividing their corresponding fractions, we get the following important intervals :—

Small Intervals and their Ratios.	Designation.
$\frac{a}{c} = \frac{135}{128}$	Chromatic semitone
$\frac{b}{c} = \frac{25}{24}$	Imperfect chromatic semitone
$\frac{a}{b} = \frac{81}{80}$	Comma
$\frac{a^2}{bc^2} = \frac{32805}{32768}$	Schisma
$\frac{a^3}{b^2 c^2} = \frac{531441}{524288}$	Comma of Pythagoras

Here we see that the large tone $\frac{9}{8}$ exceeds the diatonic semitone by the chromatic semitone $\frac{135}{128}$, and the small tone $\frac{10}{9}$ exceeds the diatonic semitone by the interval $\frac{25}{24}$. The difference of the tones $\frac{9}{8}$ and $\frac{10}{9}$ is $\frac{9}{8} \div \frac{10}{9} = \frac{81}{80}$, which is the comma. The chromatic semitone is the relation between any tone and its chromatic ♯ or ♭ : it is, in fact, the interval by which, in true transition, we pass from key to key. The imperfect chromatic semitone is a musical difference rather than a relation, it is formed by the major seventh or leading tone introduced into the minor scale. The Comma of Pythagoras is the excess of twelve fifths above seven octaves, and is the interval found at every enharmonic change of key, as from G♭ to F♯, in which the number of flats in the one signature added to that of the sharps in the other makes 12.

The mention of the Comma of Pythagoras leads us to the subject of temperament. If we try to ascertain how an instrument having only 12 fixed tones in the octave should be tuned so that all keys shall be correct, we soon find ourselves landed in a difficulty, or rather in an utter impossibility. Three major thirds must fill the octave. But three major thirds $= \frac{5}{4} \cdot \frac{5}{4} \cdot \frac{5}{4} = \frac{125}{64} = 2\left(\frac{125}{128}\right)$, which is *less than an octave.* Each major third must therefore be tuned somewhat sharp. Four minor thirds must also fill the octave. But four minor thirds $= \frac{6}{5} \cdot \frac{6}{5} \cdot \frac{6}{5} \cdot \frac{6}{5} = \frac{1296}{625} = 2\left(\frac{648}{625}\right)$, which is greater than an octave. Minor thirds are therefore tuned sensibly flat. Then as twelve fifths have to be compressed, as it were, by the Comma of Pythagoras, in order to fit into seven octaves, each fifth must be tuned *very slightly* flat. By such a process of tuning we should arrive at the equal temperament semitone, which is such an interval that a succession of twelve of them make an octave: hence the numerical expression for this artificial interval is $2^{\frac{1}{12}}$.

[A geometrical representation of the diatonic scale may be very simply obtained as follows :—

C	D	E	F	G	A	B′	B	C′
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{15}{8}$	2

Take a straight line CC′ ; let F, A be the points of trisection of CC′; and let G, E, B′, D, B be the respective middle points of CC′, CG, GC′, CE, B′C′; then if we call CC′ unity, and suppose lengths to be measured from

a point (O say) in a line with CC′ and at a unit distance to the left of C, so that OC = 1, and OC′ = 2; the distances from O of the points of division of the line CC′, as given in the figure, will be expressed by the fractions that represent the corresponding intervals from C in the diatonic scale, with the addition thereto of the harmonic seventh, B′. It is to be observed, however, that this diagram is not a pictorial representation of intervals; and that the harmonic seventh is not usually considered to belong to the musical scale.

It is right to remark here, that the foregoing article has been editorially translated into a rather more mathematical form than that in which Mr. Colin Brown sent it to us, in order the better to adapt it to the pages wherein it now appears.

The present article "On the Musical Scale," taken *first*, together with Colonel Clarke's article on "Just Intonation" (pp. 33—38 of this Volume) taken *next*, contain a short exposition of the elements of the Mathematical theory of Music, and will enable a reader to understand thoroughly the principles of Colin Brown's admirable Voice-Harmonium.]

On Two Questions in Probabilities. *By* W. S. B. Woolhouse, F.R.A.S.

Mr. Artemas Martin has solicited my opinion in regard to the two following problems, the true solutions of which have, he informs me, been much discussed in America. I give, first of all, Mr. Martin's solutions, and append thereto my own remarks in regard to the accuracy, or otherwise, of those solutions.

1. The first problem, with the Solutions thereto, is as follows:—

"A says that B says that a certain event took place; required the probability that the event did take place, p_1 and p_2 being A's and B's respective probabilities of speaking the truth."

"Mr. Todhunter has considered this question in his *Algebra* (4th ed., Art. 743, or 5th ed., Art. 755), where he states that "the event did take place, if they both speak truth, or if they both speak falsehood; and the event did not take place if only one of them speaks truth." His result therefore is $p_1p_2+(1-p_1)(1-p_2)$. But he fails to notice that 'A says that B says that an event occurred' has a different meaning from 'B told A that an event occurred and A transmitted the statement.' In the first case we are not sure that B gave A the information A says B gave him; we have only A's word for it, and the chance that he speaks truth is p_1. In the second case it is expressly stated that B gave information to A concerning the event.

Hence I conclude that the probability of the truth of the statement, when A says that B told him that the event occurred is

$$p_1[p_1p_2+(1-p_1)(1-p_2)].$$

The event did not take place if B did not tell A, the chance of which is $1-p_1$; and the probability against the event is

$$p_1[p_1(1-p_2)+p_2(1-p_1)]+(1-p_1).$$

The sum of these two probabilities is unity, as it ought to be.

In answer to a letter from me upon the subject, Mr. Todhunter says: 'In Art. 743 of the fourth edition of my *Algebra*, it is certainly assumed that B *has* given information to A respecting the event. If, however, there is any doubt on this point, the problem becomes more complex, and it seems to me that it should be treated as you suggest.'

Some mathematicians contend that the required probability is $p_1 p_2$."

According to my judgment upon the two solutions given to this problem, neither of them is correct, as extraneous and imperfect elements are improperly introduced in both. In Mr. MARTIN's solution, he observes a distinction between two specified conditions, and remarks that "A says that B says that an event occurred" has a different meaning from "B told A that an event occurred, and A transmitted the statement." But as to the latter of these conditions I have to observe that, assuming it to be known that B told A that the event occurred, the truth of the statement would in such case rest exclusively on B as an authority, and whether A transmitted it or not would have no influence whatever, and be quite immaterial as regards the enquiry.

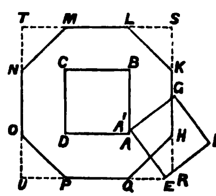
In Mr. TODHUNTER's solution, given in his *Algebra*, it is gratuitously assumed that B made one of two communications to A; viz., either that the event did or did not take place. This assumption goes quite beyond the distinct and clear enunciation of the problem, which requires us to consider simply whether B *did* or *did not* state to A that the event took place. We are not at all concerned with any vague supposition as to B having made any other statement, nor indeed as to his having necessarily made any statement at all.

According to the problem, strictly considered in the terms proposed, the probability that B *did* state to A that the event took place depends solely upon the veracity of A, and is therefore p_1; and as the probability of B speaking the truth is p_2, the required probability is incontestably $p_1 p_2$.

It is stated by Mr. MARTIN that some mathematicians in America contend that the required probability is $p_1 p_2$. For the reasons given above, I maintain that those mathematicians are undoubtedly right in their conclusions.

2. The second problem, with the Solutions thereto, is as follows:—

"If a common die be placed horizontally, at random, upon another die in a horizontal position, what is the probability that the top die will not fall off?"



"I. Let AB = 1. Place the square AEFG so that the side AG shall make an angle θ with AB. Let the square AEFG move parallel to itself so as to be exterior to the fixed square ABCD and in contact with it. The centre H will describe an octagon HKLMNOPQ, which is less than the square RSTU by the four triangles KSL, MTN, OUP and QRH, and is therefore equal to

$$(1 + \sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta,$$

or $2 + 2\sin\theta + 2\cos\theta$. The area of the square ABCD represents the number of favorable cases, and the area of the octagon the whole number of cases for the angle θ. Hence the required probability is

$$p = \int_0^{\frac{1}{2}\pi} d\theta \div \int_0^{\frac{1}{2}\pi} 2(1 + \sin\theta + \cos\theta)\, d\theta = \frac{\pi}{2\pi + 8}.$$

II. For a particular value of θ, the probability that the top die will not fall

off, is $\dfrac{1}{2(1+\sin\theta+\cos\theta)}$; hence $p = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{2(1+\sin\theta+\cos\theta)} \div \int_0^{\frac{1}{2}\pi} d\theta.$

Let $\sin\theta = \dfrac{2x}{1+x^2}$, then $\cos\theta = \dfrac{1-x^2}{1+x^2}$, $d\theta = \dfrac{2dx}{1+x^2}$,

therefore $$p = \tfrac{1}{2}\int_0^1 \frac{dx}{1+x} \div \int_0^{\frac{1}{2}\pi} d\theta = \frac{1}{\pi}\log_e 2.$$

The author of the first of the two preceding solutions of the dice problem gives as the reason why he takes his value of p, rather than the other, that all values of θ are not equally probable. In the last number of the *Analyst* it is decided that, when all values of the variable are not equally probable, the *sum* of the probabilities must be taken, which agrees with the method I have given above. But how are we to distinguish the two cases, so as to know which method to use?"

Of the two solutions to this problem, I decidedly approve of the method followed in the first, which takes the form

$$p = \frac{\Sigma_\theta \text{ favorable cases}}{\Sigma_\theta \text{ all cases}} = \frac{\text{total favorable cases}}{\text{total all cases}}.$$

This form is evidently true from first principles, and is universally applicable to all such problems. The second solution, which adopts the form

$$\Sigma_\theta \frac{\text{favorable cases}}{\text{all cases}}$$

is true only when the "all cases" in the denominator of the fraction has a value that is constant for all values of the variable; that is, in effect, when all values of the variable are equally probable. This is not so in the present problem, and the second solution is consequently erroneous.

For any particular problem in which the "all cases" in the denominator happens to be constant, the second method will obviously give the same result as the first or general method.

[In answer to a letter from the EDITOR, containing a "proof" of the foregoing article, Mr. TODHUNTER makes the following remarks:—

"My own experience leads me to the result that, in questions of probability, different solutions are frequently obtained, each correct on its own foundation, arising from different ways of understanding the question.

The present question is one which is in many books (for instance, it is on page 9 of the *Treatise* by Lubbock in the *Library of Useful Knowledge*), but I do not know where it first appeared. Since my attention has been drawn to it, I have admitted that it does not seem well expressed; but I do not think that I am quite responsible for it. The solution I think important, and the language should be improved, if possible, so as to fit closely to the solution. I think Mr. WOOLHOUSE and Mr. ARTEMAS MARTIN agree in wanting merely a condition to be stated, which is not stated. Perhaps the reference to *traditionary* testimony, which precedes the question, was held by me, or by some person who first gave the question, to justify the omission of the condition.

My *Algebra* is now stereotyped, so that changes can only be made in it to a slight extent; but when more copies are struck off, I must try to improve the matter."

See *Reprint*, Vol. XXV., pp. 18, 19, Quest. 4719; Vol. XXVI., pp. 109, 110, Quest. 4835; Vol. XXVII., pp. 27, 28, Quest. 4933.]

5161. (By the EDITOR.)—If a triangle $A'B'C'$ be polar to a given triangle ABC, with respect to a circle of given radius and variable centre (O); show that (1), when $A'B'C'$ is constant, the locus of O is a cubic hyperbola; and (2), when $A'B'C'$ is a minimum, O is the centroid of the given triangle ABC; also (3) that, in geometry of three dimensions, analogous properties subsist in regard to a tetrahedron ABCD and its inverse $A'B'C'D'$ relative to a sphere of given radius and variable centre.

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

Since, for any triangle ABC and its reciprocal $A'B'C'$ with respect to a circle of centre O and radius OR,

$$(A'B'C') = \frac{OR^4}{4} \cdot \frac{(ABC)^2}{(BOC)(COA)(AOB)},$$

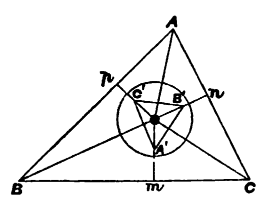
the several quantities within parentheses signifying areas [see Townsend's *Modern Geometry*, Art. 180, No. 3°]; hence, ABC being supposed fixed and O variable, if $(A'B'C')$ be constant, then is the continued product (BCO) (COA) (AOB) constant, which manifestly proves (1); and if $(A'B'C')$ be a minimum, then is the aforesaid product a maximum, and the point O consequently the centroid of the area (ABC), which proves (2).

Since analogously, in geometry of three dimensions, for any tetrahedron ABCD and its reciprocal $A'B'C'D'$ with respect to a sphere of centre O and radius OR,

$$(A'B'C'D') = \frac{OR^6}{36} \cdot \frac{(ABCD)^3}{(BCDO)(CDAO)(DABO)(ABCO)},$$

the several quantities within parentheses signifying volumes [see *Reprint*, Vol. XIII., pp. 71, 72]; it follows, in precisely the same manner, that if $(A'B'C'D')$ be constant, then is the continued product (BCDO) (CDAO) (DABO) (ABCO) constant, and therefore the locus of O a quartic hyperboloid; and if $(A'B'C'D')$ be a minimum, then is the aforesaid product a maximum, and the point O consequently the centroid of the volume (ABCD).

II. *Solution by the* PROPOSER.



Let $A'B'C'$ be the triangle polar to the given triangle ABC, relative to the circle whose centre is O and radius r; then, if $OA'm$, $OB'n$, $OC'p$ be drawn, these lines will be respectively perpendicular to BC, CA, AB, and $r^2 = OA'.Om = OB'.On = OC'.Op$.

Now we have

$$\Delta A'OB' : \Delta ABC = OA'.OB' : AC.BC$$
$$= r^4 : AC.BC.Om.On;$$

hence, putting A, A_1, A_2, A_3, Δ, for the areas of the triangles ABC, BOC, AOC, AOB, $A'B'C'$, we shall have

$$A : A'OB' = 4A_1A_2A_3 : r^4A_3;$$
$$A : A'OC' = 4A_1A_2A_3 : r^4A_2;$$
$$A : B'OC' = 4A_1A_2A_3 : r^4A_1;$$

therefore $\quad A : \Delta = 4A_1A_2A_3 : r^4A$, and $\Delta = \dfrac{r^4A^2}{4A_1A_2A_3}$.

But $A_1 + A_2 + A_3 = A$, therefore $A_1A_2A_3$ is a maximum when $A_1 = A_2 = A_3 = \frac{1}{3}A$; hence Δ (or A′B′C′) is a minimum $\left(= \dfrac{27r^4}{4A}\right)$ when O is the centroid of the given triangle ABC.

When the area of the polar triangle A′B′C is constant, then $Om \,.\, On \,.\, Op$ will be constant, and the locus of O will be a cubic hyperbola.

The proof is precisely similar in geometry of three dimensions.

5208. (By Professor SYLVESTER, F.R.S.)—Let the *magnitude* of any ramification signify the number of its branches, and let its partial magnitudes in respect to any node signify the magnitudes of the ramifications which come together at that node. If at any node the largest magnitude exceeds by k the sum of the other magnitudes, let the node be called superior by k, or be said to be of superiority k; but if no magnitude exceeds the sum of the other magnitudes, let the node be called subequal. Then the theorem is, in any ramification, *either* there is one and only one subequal node; *or else* there are two and only two nodes each superior by unity, these two nodes being contiguous.

[Professor SYLVESTER remarks that this remarkable theorem in the pure theory of Colligation, or Cause and Effect, is due to the transcendent genius of M. Camille Jordan; and that it is worthy of notice that an infinite ramification serves to express the possibility of Time, or the natural order of consecution of groups of phenomena, being variable at will, by varying the position of the origin or first cause; and thus greatly extends the common conception of time as a determinate linear order of sequence of such groups. For clearness sake, the original form of enunciation of the Question has been somewhat altered by Professor CAYLEY.]

Solution by Professor CAYLEY, F.R.S.

The proof consists in showing that (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity; and if there is one such node, then there is, contiguous to it, another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is either of smaller superiority, or else subequal; for, these theorems holding good, we can, by (3), always arrive at a node which is either subequal or else superior by unity; in the former case, by (1), the subequal node thus arrived at is unique; in the latter case, by (2), we have, contiguous to the node arrived at, a second node superior by unity; and we have thus a unique pair of nodes each superior by unity.

I will prove only (3), as it is easy to see that the like process applies to the proof of (1) and (2).

Let the whole magnitude be n; and suppose at a node P which is superior by k, the largest magnitude is a, and that the other magnitudes

are, say, β, γ, δ. We have $\alpha = \beta+\gamma+\delta+k$; and since $n = \alpha+\beta+\gamma+\delta$, we have thence $n = 2\alpha-k$, or $\alpha = \frac{1}{2}(n+k)$, $\beta+\gamma+\delta = \frac{1}{2}(n-k)$: clearly k is even or odd according as n is even or odd.

Suppose now that we pass from P, along the branch of magnitude α, to a contiguous node Q; and let the magnitudes for Q be $\alpha', \beta', \gamma', \delta', \epsilon'$, where α' denotes the magnitude for the branch QP. We have $\alpha' = \beta+\gamma+\delta+1$, for the ramification consists of the branch QP and of the ramifications of magnitudes β, γ, δ which met in P. We have thus

$$\alpha' = \tfrac{1}{2}(n-k)+1 = \tfrac{1}{2}n - \tfrac{1}{2}(k-2);$$

and thence $$\beta'+\gamma'+\delta'+\epsilon' = \tfrac{1}{2}n+\tfrac{1}{2}(k-2).$$

Supposing here that k is greater than 1, viz. that it is $=$ or > 2, $k-2$ is 0 or positive; and if α' be the greatest magnitude belonging to the node Q, this is a subequal node. But it may be that α' is not the greatest magnitude; suppose then that the greatest magnitude is β', we have

$$\beta' = \tfrac{1}{2}n+\tfrac{1}{2}(k-2)-\gamma'-\delta'-\epsilon',$$
$$\alpha'+\gamma'+\delta'+\epsilon' = \tfrac{1}{2}n-\tfrac{1}{2}(k-2)+\gamma'+\delta'+\epsilon',$$

and thence $$\beta'-(\alpha'+\gamma'+\delta'+\epsilon') = k-2-2(\gamma'+\delta'+\epsilon');$$

viz., either the node is subequal, or else, being superior, the superiority is at most $= k-2$; that is, if from the node P, of superiority $=$ or > 2, we pass along the branch of greatest magnitude to the contiguous node Q, this is either subequal, or else of superiority less than that of P; which is the foregoing proposition (3).

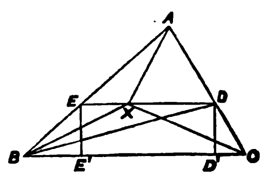
The subequal node, and the two nodes of superiority 1, in the cases where they respectively exist, may be termed the centre and the bicentre respectively; and the theorem thus is, every ramification has either a centre or else a bicentre. But the centre and bicentre here considered, due (as remarked by Professor SYLVESTER) to M. CAMILLE JORDAN, and which may for distinction be termed the centre and bicentre of *magnitude*, are quite distinct from the centre and bicentre discovered by Professor SYLVESTER, and considered in my researches upon trees (*British Association Report*, 1875). These last may for distinction be termed the centre and bicentre of *distance*, viz., we here consider, not the magnitude, but the length of a ramification, such length being measured by the number of branches to be travelled over in order to reach the most distant terminal node. The ramification has either a centre or else a bicentre of distance, viz., the centre is a node such that for two or more of the ramifications which proceed from it the lengths are equal and superior to those of the other ramifications (if any); the bicentre a pair of contiguous nodes such that (disregarding the branch which unites the two nodes) there are from the two nodes respectively (one at least from each of them) two or more ramifications the lengths of which are equal to each other and superior to those of the other ramifications (if any).

It is very noticeable how close the agreement is between the proofs for the existence of the two kinds of centre or bicentre respectively. Say, first as regards distance, if at any node the length of the longest branch exceeds by k the length of the next longest branch or branches; then, the node is superior by k, or is of the superiority k; but, if there are two or more longest branches, then the node is subequal. And say next, in regard to magnitude, if at any node the largest magnitude exceeds by k the sum of all the other magnitudes, the node is superior by k, or has a superiority k; but if the largest magnitude does not exceed the sum of the other magnitudes, then the node is subequal. Then, whether we

attend to distance or to magnitude, the three propositions hold good: (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity; and if there is one such node, there is contiguous to it another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is of smaller superiority or else subequal; whence, as in the solution just referred to, there is always, as regards distance, a centre or bicentre; and there is always, as regards magnitude, a centre or bicentre.

5213. (By Professor CROFTON, F.R.S.)—If three points X, Y, Z are taken at random within a triangle ABC, show [as may be done without integration] that it is an even chance that both Y and Z lie on some one of the triangles XBC, XCA, XAB.

I. *Solution by* Colonel A. R. CLARKE, C.B., F.R.S.



The number of cases in which both Y and Z lie in some one of the triangles XBC, XCA, XAB is $\frac{1}{4}(a^2u^2+b^2v^2+c^2w^2)$, where u, v, w are the perpendiculars from X on a, b, c respectively. Now, to get the sum of the quantities u^2 for every position of X, draw EXD parallel to BC, and EE′, DD′ perpendicular to BC. Then the sum of the quantities u^2 for all positions of X along ED is EE′ × rectangle EDD′E′. Now suppose this rectangle to turn round ED until it becomes perpendicular to the triangle ABC ($=\Delta$); then, as ED moves from BC to A, the rectangle generates a tetrahedron of volume $=\frac{2}{3}\frac{\Delta^2}{a}$, and the sum of the quantities u^2 is this volume multiplied by the distance of its centroid from a plane through BC perpendicular to ABC. This distance is $=\frac{\Delta}{a}$, and hence for all positions of X the sum of the quantities $\frac{1}{4}(a^2u^2+b^2v^2+c^2w^2)$ is evidently $\frac{1}{2}\Delta^3$. But Δ^3 is the total number of ways in which the three points may be taken; so that, as stated in the Question, the chance is $\frac{1}{2}$.

II. *Solution by* ELIZABETH BLACKWOOD.

Let h be the perpendicular height of A from BC (see figure to first solution); then the chance that the height of X above BC lies between x and $x+dx$ is $\frac{2(h-x)}{h^2}\,dx$. Given that X satisfies this condition, the chance that Y and Z both fall in the triangle BXC is $\left(\frac{\Delta\,\text{BXC}}{\Delta\,\text{ABC}}\right)^2=\frac{x^2}{h^2}$; therefore the chance that, when X is taken at random, Y and Z both fall in the

triangle BXC, is $$\int_0^h \frac{x^2}{h^2} \cdot \frac{2(h-x)}{h^2}\,dx = \tfrac{1}{6}.$$

The chance for each of the triangles CXA, AXB is also, by the same reasoning $\frac{1}{6}$; hence the required chance is $\frac{3}{6}$ or $\frac{1}{2}$.

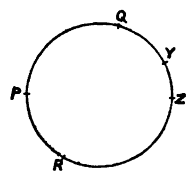
III. *Solution by* Hugh McColl, B.A.

Definition.—Any magnitude m is divided into three random parts a, b, c when for every given value of any part a all values of b or c between 0 and $m-a$ are equally probable.

If we can prove that the three triangles AXB, AXC, BXC are random portions of the triangle ABC according to the above definition, Prof. Crofton's proposition will become a particular case of the following more general one :—

If any magnitude m be divided into three random parts, and two random points Y, Z be taken in the magnitude m, it is an even chance that both Y and Z are in some one of the three random parts which make up the given magnitude.

We will first prove the more general proposition. It is evident, *à priori*, that the chance is the same for all species of magnitude. For the sake of symmetry, let the magnitude m be the circumference of a circle. Let P be a fixed point, and Q, R two random points in this circumference. Then, starting from the point P and measuring round the circumference *in one continuous direction—which we shall suppose throughout to be that in which the hands of a clock move*—the three random segments of the circumference will be PQ, QR, RP, or PR, RQ, QP, according as Q or R comes next to P.



Let Y, Z be two other random points in the circumference. Since all positions on the circumference are equally probable, for each of the four random points Q, R, Y, Z, all permutations of these four letters are equally probable. Let $q =$ the chance that one particular permutation, as QYZR, will take place. Then (always taking P as the starting point), $q =$ the chance that the order of the letters will be PQYZR, and $q\,\underline{|4} = 1$ for all the $\underline{|4}$, or 24 permutations of the letters QRYZ are equally probable, mutually exclusive, and one *must* happen. When Q comes before R, and Y before Z, the favourable permutations are PYZQR, PQYZR, PQRYZ. By exchanging Q and R in these three, we get three more favourable permutations. If in each of these six permutations we exchange Y and Z, we shall have in all twelve favourable permutations. The required chance is therefore $12q$. But it has been shewn that $24q = 1$; therefore $12q = \frac{1}{2} =$ required chance.

It remains to be shown that the triangles AXB, AXC, BXC (see Figure to first Solution) are random divisions of the whole triangle ABC.

Let the area of any of the three triangles, as BXC, be known or given. Then the perpendicular distance of X from BC is given, and X may have any position in a line DE parallel to BC. As all positions of X between D and E are equally probable, all values of the area of the triangle AXB between zero and the area of the triangle AEB (which = triangle ABC − triangle BXC) are equally probable. Hence the triangles AXB, AXC, BXC are three random parts of the triangle ABC, and by the preceding proposition the chance required is $\frac{1}{2}$.

IV. *Solution by* R. E. Riley, B.A.

Consider the two triangles ABX, CBX; it is clear that there are four arrangements possible, viz.,

Y in ABX, Z in BXC .... (1),
Z in ABX, Y in BXC .... (2),
Y, Z in ABX, Y, Z in BXC .... (3, 4).

Hence, as far as this pair of triangles is concerned, the proposed event occurs in two out of the four arrangements, and the chance is even; and the same reasoning being applicable to the other two pairs of triangles, on the whole there is an even chance of the proposed event.

5231. (By Hugh McColl, B.A.)—Two points, P and Q, are successively taken at random within a given sphere; show that the chance that the sphere of which P is the centre and PQ the radius lies wholly within the given sphere is $\frac{1}{20}$.

Solution by Elizabeth Blackwood.

Imagine an infinite number of concentric spheres whose radii are dx, $2dx$, $3dx$, ... $a-dx$, a, the outer sphere (with radius a) being the given sphere. The chance that the point P falls without the sphere whose radius is x, and within the sphere whose radius is $x+dx$, is $\dfrac{\frac{4}{3}\pi(x+dx)^3-\frac{4}{3}\pi x^3}{\frac{4}{3}\pi a^3}$, which $=\dfrac{3x^2dx}{a^3}$. Given that P occupies such a position, the point Q must (for all favourable cases) be within a sphere of radius $a-x$, which touches the given sphere; and the chance of this, for all positions of P, as long as x remains constant, is $\frac{4}{3}\pi(a-x)^3 \div \frac{4}{3}\pi a^3 = (a-x)^3 \div a^3$.
The required chance is therefore

$$\int_0^a \frac{(a-x)^3}{a^3}\cdot\frac{3x^2}{a^3}\,dx = \frac{3}{a^6}\int_0^a x^2(a-x)^3\,dx = \frac{1}{20}.$$

5222. (By the Editor.)—If P be the orthocentre of a triangle; O, I, I_1, I_2, I_3 the centres, and R, r, r_1, r_2, r_3 the radii, of the circumscribed, inscribed, and escribed circles of the triangle; and ρ the radius of the circle inscribed in the orthocentric triangle; prove that

$$PI^2 = 2(r^2 - RP),\quad PI_1^2 = 2(r_1^2 - R\rho),\quad PI_2^2 = 2(r_2^2 - R\rho),$$
$$PI_3^2 = 2(r_3^2 - R\rho),\quad PO^2 = R^2 - 4R\rho.$$

I. *Solution by* L. W. JONES, B.A.

This is readily proved from the condition that it may be possible to describe triangles circumscribing a circle whose radius is r, and self-conjugate with regard to a circle whose radius is σ; viz.,

$$2r^2+\sigma^2 = \delta^2 = \text{square of distance between centres.}$$

The square of the radius of the circle with regard to which the triangle is self-conjugate is $-4R^2 \cos A \cos B \cos C$, therefore

$$(PI)^2 = 2r^2-4R^2 \cos A \cos B \cos C = 2(r^2-R\rho).$$

Similarly, we have

$$2r^2-PI^2 = 2r_1^2-PI_1^2 = \&c. = 4R^2 \cos A \cos B \cos C = 2R\rho,$$

or $\quad (PI_1)^2 = 2(r_1^2-R\rho), \quad (PI_2)^2 = 2(r_2^2-R\rho), \quad (PI_3)^2 = 2(r_3^2-R\rho).$

The same method gives

$$(PO)^2 = 2\sigma^2+R^2 = R^2-8R^2 \cos A \cos B \cos C,$$

where O is the centre of the circumscribed circle.

[This value of $(PO)^2$ is otherwise investigated in Dr. BOOTH's *New Geometrical Methods*, Vol. II., p. 302, eq. (b), and shown to be reducible to the forms $R(R-4\rho)$, $9R^2-a^2-b^2-c^2$.]

II. *Solution by the* EDITOR.

By formulæ investigated in the recently published volume of Dr. BOOTH's *New Geometrical Methods* [see Vol. II., p. 295, eq. (j), and p. 301, eq. (c)] we have $\quad \rho = 2R \cos A \cos B \cos C = \dfrac{s^2-(2R+r)^2}{2R}$;

hence the expression for $(PI)^2$, given on p. 305, eq. (c) of the same volume, becomes

$$(PI)^2 = 4R^2+4Rr+3r^2-s^2 = 2r^2-\left\{s^2-(2R+r)^2\right\} = 2(r^2-R\rho).$$

In like manner may the three formulæ given in group (d) on p. 305 of Dr. BOOTH's second volume be reduced to the other forms given in the question. As an algebraical exercise, however, these reductions are proposed as Question 5312.

III. *Solution by the* PROPOSER; Prof. EVANS, M.A.; E. RUTTER; *and others.*

In the triangle PAI we have $AI = \dfrac{r}{\sin\frac{1}{2}A}$; $AP = 2R \cos A$, $PAI = \frac{1}{2}(C-B)$; hence

$$(PI)^2 = 4R^2 \cos^2 A + \frac{r^2}{\sin^2\frac{1}{2}A} - \frac{4Rr \cos A}{\sin\frac{1}{2}A} \cos\tfrac{1}{2}(C-B).$$

Now we have $$\frac{r^2}{\sin^2\frac{1}{2}A}-\frac{4Rr\cos A}{\sin\frac{1}{2}A}\cos\tfrac{1}{2}(C-B)$$

$$=\frac{4Rr\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A}-\frac{4Rr}{\sin\frac{1}{2}A}\cos\tfrac{1}{2}(C-B)(1-2\sin^2\tfrac{1}{2}A)$$

$$=-\frac{4Rr\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A}+8Rr\cos\tfrac{1}{2}(C-B)\cos\tfrac{1}{2}(C+B)$$

$$=-\frac{4Rrs}{a}+4Rr(\cos B+\cos C)=-bc+4Rr(\cos B+\cos C),$$

hence $\quad (PI)^2 = 4R^2+4Rr(\cos B+\cos C)-a^2-bc.$

Similarly, $\quad (PI)^2 = 4R^2+4Rr(\cos C+\cos A)-b^2-ca,$

and $\quad (PI)^2 = 4R^2+4Rr(\cos A+\cos B)-c^2-ab;$

hence $$3(PI)^2 = 12R^2+8Rr(\cos A+\cos B+\cos C) -(a^2+b^2+c^2+ab+bc+ca);$$

but $$R(\cos A+\cos B+\cos C) = R+r,$$
$$a^2+b^2+c^2+ab+bc+ca = 3S^2-4Rr-r^2,$$

therefore $(PI)^2 = 4R^2+4Rr+3r^2-s^2$ [which agrees with Dr. BOOTH's result]. Now

$$\rho = 2R(\cos A\cos B\cos C) = R(\sin^2 A+\sin^2 B+\sin^2 C-2),$$

therefore $\quad 4R\rho = a^2+b^2+c^2-8R^2 = 2s^2-2r^2-8Rr-8R^2;$

therefore $\quad (PI)^2+2R\rho = 2R^2, \text{ or } (PI)^2 = 2(r^2-R\rho).$

5159. (By Dr. BOOTH, F.R.S.)—Given the sum of the focal vectors of a parabola, and the chord that joins the ends of these vectors; find the area of the parabolic sector whose vertex is the focus.

I. *Solution by* R. TUCKER, M.A.,

Let SP (r,α), SQ (r',β) be the two vectors; then we have

$$r = a\sec^2\tfrac{1}{2}\alpha,\quad r' = a\sec^2\tfrac{1}{2}\beta;$$

therefore $$A = \tfrac{1}{2}\int_\alpha^\beta r^2\,d\theta = \tfrac{1}{2}a^2\int_\alpha^\beta \sec^4\tfrac{1}{2}\theta\,d\theta$$

$$= a^2\left[(\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha)+\tfrac{1}{3}(\tan^3\tfrac{1}{2}\beta-\tan^3\tfrac{1}{2}\alpha)\right].$$

Let $$e = r+r' = a(\sec^2\tfrac{1}{2}\alpha+\sec^2\tfrac{1}{2}\beta),$$

$$d^2 = PQ^2 = r^2+r'^2-2rr'\cos(\beta-\alpha) = (r+r')^2-4rr'\cos^2\tfrac{1}{2}(\beta-\alpha);$$

that is, $\quad e^2-d^2 = 4a^2\cos^2\tfrac{1}{2}(\beta-\alpha)\sec^2\tfrac{1}{2}\alpha\sec^2\tfrac{1}{2}\beta = \lambda^2;$

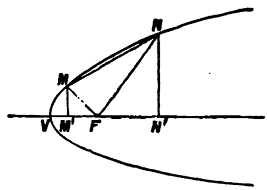
therefore $\lambda = 2a\,(1 + \tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\beta)\ \tan^2\tfrac{1}{2}\alpha + \tan^2\tfrac{1}{2}\beta = \dfrac{c}{a} - 2;$

therefore
$$\tan\tfrac{1}{2}\beta - \tan\tfrac{1}{2}\alpha = \left(\frac{c-\lambda}{a}\right)^{\frac{1}{2}},$$

$$\tan^3\tfrac{1}{2}\beta - \tan^3\tfrac{1}{2}\alpha = \left(\frac{c-\lambda}{a}\right)^{\frac{1}{2}}\left(\frac{2c+\lambda-6a}{2a}\right).$$

Hence
$$A = a^2\left(\frac{c-\lambda}{a}\right)^{\frac{1}{2}}\left(\frac{2c+\lambda}{6a}\right) = \tfrac{1}{6}(2c+\lambda)\,[a(c-\lambda)]^{\frac{1}{2}}.$$

II. *Solution by* Professor F. Armenante.



Let $MN = c$, $MF + FN = s$, and let $y^2 = px$ be the equation of the parabola. The lengths of focal vectors which pass through the points $M(x', y')$, $N(x'', y'')$ will be respectively
$$x' + \tfrac{1}{4}p,\quad x'' + \tfrac{1}{4}p;$$
hence
$$s = x' + x'' + \tfrac{1}{2}p \text{ .... } (1).$$
We have also
$$c^2 = (x''-x')^2 + (y''-y')^2 = x''^2 + x'^2 - 2x''x' + y''^2 + y'^2 - 2y''y'$$
$$= s^2 - \tfrac{1}{4}p^2 - 4x''x' - 2p\,(x''x')^{\frac{1}{2}} \text{ .... } (2)$$
because the points (x', y'), (x'', y'') are taken in the parabola, and $x'' + x' = s - \tfrac{1}{2}p$.

Now, by Archimedes' theorem, we have
$$6\text{VNN}' = 4x''y'' = \frac{4y''^3}{p};\quad 6\text{VMM}' = \frac{4y'^3}{p};$$
and also
$$6\Delta\text{NN}'\text{F} = 3y''\left(\frac{y''^2}{p} - \tfrac{1}{4}p\right);\quad 6\Delta\text{MM}'\text{F} = 3y'\left(\tfrac{1}{4}p - \frac{y'^2}{p}\right);$$
therefore
$$6 \text{ parabolic sector MFN} = \frac{4y''^3}{p} - \frac{4y'^3}{p} - 3y''\left(\frac{y''^2}{p} - \tfrac{1}{4}p\right) - 3y'\left(\tfrac{1}{4}p - \frac{y'^2}{p}\right)$$
$$= (y''-y')\left(\frac{y''^2 + y'^2 + y''y'}{p} + \tfrac{3}{4}p\right) = (y''-y')\left(\frac{px'' + px' + p\,(x''x')^{\frac{1}{2}}}{p} + \tfrac{3}{4}p\right),$$
$$= (y''-y')\,[s + \tfrac{1}{4}p + (x''x')^{\frac{1}{2}}], \text{ by } (1).$$

But $(y''-y')^2 = p\,(x''-x') - 2p\,(x''x')^{\frac{1}{2}} = p\,[s - \tfrac{1}{2}p - 2\,(x''x')^{\frac{1}{2}}];$

$\therefore\quad 6 \text{ parabolic sector} = [s + \tfrac{1}{4}p + (x''x')^{\frac{1}{2}}]\ \{p\,[s - \tfrac{1}{2}p - 2\,(x''x')^{\frac{1}{2}}]\}^{\frac{1}{2}}.$

Putting herein the value of $(x''x')^{\frac{1}{2}}$ obtained from equation (2), we obtain
$$\text{parabolic sector} = \tfrac{1}{8}\,[s \mp (s^2-c^2)^{\frac{1}{2}}]\ \{p\,[s \pm \tfrac{1}{2}(s^2-c^2)\}^{\frac{1}{2}}.$$

5227. (By B. WILLIAMSON, M.A.)—If S denote the surface, and V the volume, of the cone standing on the focal ellipse of an ellipsoid, and having its vertex at an umbilic; prove that

$$S = \pi a (b^2-c^2)^{\frac{1}{2}}, \quad V = \tfrac{1}{3}\pi c (b^2-c^2),$$

where a, b, c are the principal semiaxes of the ellipsoid.

Solution by Professor TOWNSEND, M.A., F.R.S.

Denoting by E the area of the focal ellipse, by z the ordinate of the umbilic perpendicular to its plane, by r_1 and r_2 the distances of the umbilic from the extremities of its major axis, by p_1 and p_2 the perpendiculars from the same extremities upon the tangent plane at the umbilic, and by θ and ϕ the angles made with that plane by the area of the focal ellipse and by the surface of the cone subtending it from the umbilic; then,

since $\quad E = \pi (a^2-c^2)^{\frac{1}{2}} (b^2-c^2)^{\frac{1}{2}}, \quad z = c\left(\dfrac{b^2-c^2}{a^2-c^2}\right)^{\frac{1}{2}}, \quad$ and $\quad V = \tfrac{1}{3}Ez,$

therefore &c. as regards V; and since again

$$\cos\theta = \frac{a}{b}\left(\frac{b^2-c^2}{a^2-c^2}\right)^{\frac{1}{2}}, \qquad \sec\phi = \left(\frac{r_1 r_2}{r_1 r_2 - p_1 p_2}\right)^{\frac{1}{2}} = \frac{b}{(b^2-c^2)^{\frac{1}{2}}},$$

and $\quad S = E \cos\theta \sec\phi, \quad$ therefore &c. as regards S.

ON MR. ARTEMAS MARTIN'S FIRST QUESTION IN PROBABILITIES.

By Professor CAYLEY, F.R.S.

The question was, "A says that B says that a certain event took place: required the probability that the event did take place, p_1 and p_2 being A's and B's respective probabilities of speaking the truth."

The solutions referred to or given [on pp. 77—79 of this volume of the *Reprint*] give the following values for the probability in question:—

TODHUNTER'S *Algebra* .... $p_1p_2 + (1-p_1)(1-p_2)$.
ARTEMAS MARTIN .... $p_1[p_1p_2 + (1-p_1)(1-p_2)]$.
American Mathematicians and WOOLHOUSE ... p_1p_2.

It seems to me that the true answer cannot be expressed in terms of only p_1, p_2, but that it involves two other constants β and k; and my value is—

CAYLEY .... $p_1p_2 + \beta(1-p_1)(1-p_2) + k(1-\beta)(1-p_1)$.

In obtaining this I introduce, *but I think of necessity*, elements which Mr. WOOLHOUSE calls extraneous and imperfect.

B told A that the event happened, or he did not tell A this; the only evidence is A's statement that B told him that the event happened; and the chances are p_1 and $1-p_1$. But, in the latter case, either B told A that the event did not happen, or he did not tell him at all; the chances (on the supposition of the incorrectness of A's statement) are β and $1-\beta$;

and the chances of the three cases are thus p_1, $\beta(1-p_1)$, and $(1-\beta)(1-p_1)$. On the suppositions of the first and second cases respectively, the chances for the event having happened are p_2 and $1-p_2$; on the supposition of the third case (viz., here there is no information as to the event) the chance is k, the antecedent probability; and the whole chance in favour of the event is

$$p_1p_2+\beta(1-p_1)(1-p_2)+k(1-\beta)(1-p_1).$$

If $\beta=1$, we have TODHUNTER's solution; if $\beta=0$, and also $k=0$, we have the solution preferred by WOOLHOUSE; but we do not (otherwise than by establishing between k and β a relation which is quite arbitrary) obtain MARTIN's solution. The error in this seems to be as follows:—A says that B told him as to the event, and says further that B told him that the event did happen; probability of the truth of the compound statement is taken to be $=p_1{}^2$; whereas, in calling the probability of A's speaking the truth p_1, we mean that if A makes the statement, "B says that the event took place," this is to be regarded as a simple statement, and the probability of the truth of the statement is $=p_1$; viz., I think that MARTIN introduces into his solution a hypothesis contradictory to the assumptions of the question.

I remark further that in my solution I assume that the event is of such a nature that, when there is *any* testimony in regard to it, the probability is determined by that testimony, irrespectively of the antecedent probability. This is quite consistent with the antecedent probability being, not zero, but as small as we please; so that, if k is (as it may very well be) indefinitely small, the whole probability is the same as if k were $=0$. But there is absolutely no reason for assigning any determinate value to β; so that the solutions $p_1p_2+(1-p_1)(1-p_2)$ and p_1p_2, which assume respectively $\beta=1$ and $\beta=0$, seem to me on this ground erroneous.

ON MR. ARTEMAS MARTIN'S FIRST QUESTION IN PROBABILITIES.

By W. S. B. WOOLHOUSE, F.R.A.S.

Professor CAYLEY having obligingly sent his foregoing note on the above for my perusal, I am enabled to make the following additional remarks appertaining to the same:—

As Professor CAYLEY materially alters both the form and substance of the inquiry by the introduction of fictitious elements of supposed unknown value, I cannot perceive the fitness of his statement as regards the particular question under consideration, the enunciation of which is so simple, clear, and decided. In effecting a solution, the first step in the process naturally relates to the subsistence of the specific proposition, "B told A that the event took place," or that of the general negative proposition, viz., B did not tell A that the event took place. In the latter case, Professor CAYLEY observes that, on the supposition of the incorrectness of A's statement, "either B told A that the event did not happen, or he did not tell him at all;" and he denotes the respective chances by β and $1-\beta$.

But here I have to remark that the question gives no information respecting β, and that its value must necessarily be regarded as *zero*, since it represents the probability of the truth of an accurately defined supposition that is not supported by the slightest positive testimony.

In fact, it is obvious that the general negative proposition before mentioned admits of being divided or resolved into an indefinite number of independent, distinct, and equally defined suppositions, the separate probabilities of which are inappreciable; and the case that "B told A that the event did not happen" is only one of them.

The real purport of the argument advanced in this note is, in fact, that the question gives no information respecting β and k, and that, in the absence of such information, these chances are each $= 0$, not as an assumption, but absolutely in the nature of the enquiry.

On Mr. Artemas Martin's First Question in Probabilities. *By* Septimus Tebay, B.A.

Mr. Woolhouse's view of this problem is perfectly correct. The problem is free from antecedent hypothesis, and depends solely on the positive statements of A and B. For instance, A says that B says he has been reading the *Times;* required the probability that he has read the *Times.* This construction excludes antecedent contingencies; and whether he has read the paper or not, the probability that he has read it is $p_1 p_2$.

If the event itself hangs upon probability, p_1 and p_2 may be very inconsistent. Let $p_1 = \frac{2}{3}$, $p_2 = \frac{4}{5}$, and suppose A says that B says he has thrown *ace* with a single die; required the probability that *ace* was thrown. The answer is plainly $\frac{1}{6}$, and not $\frac{8}{15}$.

5112. (By Prof. Townsend, F.R.S.) — Show that every bicircular quartic curve in a plane, having a concyclic tetrad of real and distinct foci, is the stereographic projection, from the opposite point of a sphere touching the plane at the centre of negative inversion of the curve, of the complete intersection of the sphere with a quadric cone diverging from its centre; the four tetrads of concyclic foci of the plane being the projections of the four tetrads of complanar foci of the spherical curve, and the four circles of inversion of the former being those of the four circles of symmetry of the latter.

Solution by the Proposer.

Denoting by A, B, C, D the four real and distinct foci of the curve, by O the centre of the circle on which they lie, and by P, Q, R their three centres of homology, two of which P and Q are necessarily external to the circle, while the third R is necessarily internal to it; then, as is well known (See Casey's *Bicircular Quartics*, Art. 25), are O, P, Q, and R the four centres of inversion of the curve, the constant rectangle of inversion being positive for the three former, O, P, and Q, and negative for the

latter, R. Hence, conceiving a perpendicular RS erected to the plane of the curve at R, the square of which is equal in magnitude and opposite in sign to the constant rectangle of inversion for R, and a sphere described on RS as diameter, it is evident that every pair of inverse points X and Y of the curve with respect to R will lie in a plane with RS and subtend a right angle at S, and will consequently be the projections from S of a pair of diametrically opposite points X′ and Y′ on the sphere; therefore, the complete sphero-quartic, in which the cone subtending the curve from any vertex would intersect any sphere passing through the vertex and having its centre on the perpendicular from it to the plane, consists, in this case, of a pair of opposite twin curves on the sphere having RS for diameter, both of which are consequently sphero-conics on its surface, and therefore &c. as regards the first part of the property; and the second part follows immediately from the known properties that every focus of a curve, plane or spherical, is the evanescent limit to a circle, of the plane or sphere, having double imaginary contact with the curve, and that, in stereographic projection, every circle of the sphere projects into a circle of the plane, and conversely.

The converse of the above property—viz., that every sphero-conic twin pair projects stereographically, from any point on the sphere upon the tangent plane at the opposite point, into a bicircular quartic, whose four tetrads of concylic foci, with the four circles of inversion on which they lie, are the projections of the four complanar tetrads of the pair, with those of the four circles of symmetry on which they lie—is manifestly a particular case of the more general and obvious property, that the complete quartic of intersection of any two quadric surfaces projects so from any umbilical point of either surface, or of any third quadric of the entire system passing through it, upon the tangent plane at the opposite umbilic of the same surface.

In the particular case of the sphero-conic twin pair, the quartic of projection is evidently a Cassinian when the projection is from a point on the great circle containing the four real foci of the pair, and a Cartesian when it is from one of the four foci themselves.

The magnitudes of all angles remaining unchanged in stereographic projection, the two properties 4795 and 4844, already found on other principles [*Reprint*, Vol. XXV. pp. 17, 42], follow immediately, by virtue of the above, from the known property that the two great circles connecting any point of a sphero-conic with the two pairs of opposite foci make equal angles with the curve at the point.

5131. (By Professor SYLVESTER, F.R.S.)—When a pencil of rays falling on a curved surface are each *deflected through a constant angle*, the curve which the deflected rays envelop may be called the Caustic of Deflexion. Suppose now that a pencil of rays issuing from a fixed point fall upon a circle; prove (1) that the caustic of deflexion is a conic, find its equation in rectangular coordinates, and show that its discriminant is a perfect square. Prove also (2) that when the centre of the incident pencil approaches indefinitely near to the circle, the limiting form of the caustic of deflexion is a pair of coincident straight lines; and again, that when the

angle of deflexion approaches indefinitely near to zero, the limiting form of the caustic is a pair of crossing straight lines, which are real if the centre of the pencil is inside, and imaginary if it is outside the circle; the ultimate as distinguished from the limiting forms of the caustic in each of the cases supposed being evidently a single point. Explain also what happens when the tangent of the angle of deflexion is $\sqrt{(-1)}$.

5232. (By J. C. MALET, M.A.)—For a bicircular quartic with a finite node, prove that the caustic of deflexion for a pencil of rays from the node is a conic.

Solution by C. LEUDESDORF, M.A.

Taking the radiant point as origin, it is easily seen that the ray incident at (x', y') on the deflecting surface is, after deflection through an angle α, directed along the line

$$x(y' + x' \tan\alpha) + y(y' \tan\alpha - x') = \tan\alpha\,(x'^2 + y'^2).$$

Turning the axes through an angle α, this becomes

$$xx' + yy' = \sin\alpha\,(x'^2 + y'^2) \quad\text{..............................(1).}$$

Hence (as in SALMON'S *Higher Plane Curves*, § 121) we may find the envelope of (1) from the equation to the parallel curve (at distance k) by writing $k^2 = x^2 + y^2$, and putting $\dfrac{x}{2\sin\alpha}$, $\dfrac{y}{2\sin\alpha}$ for x, y respectively.

For the circle, centre $(\lambda, 0)$, radius unity, the parallel curves are

$$(x-\lambda)^2 + y^2 = (1 \pm k)^2;$$

therefore the caustic of deflection is

$$(x - 2\lambda\sin\alpha)^2 + y^2 = \left\{2\sin\alpha \pm (x^2+y^2)^{\frac{1}{2}}\right\}^2,$$

which reduces to $\qquad (1-\lambda^2)\dfrac{\sin\alpha}{r} = \pm 1 - \lambda\cos\theta,$

the equation of a conic: agreeing with Professor TANNER'S solution of Quest. 5131 [*Reprint*, Vol. XXVII., p. 43].

Otherwise:—Since (1) may be written $xx'' + yy'' = 1$, where x'', y'' are the coordinates of a point on the inverse curve of the deflecting surface with regard to the origin (the constant of inversion being $\operatorname{cosec}^{\frac{1}{2}}\alpha$), therefore the tangential (in Dr. BOOTH'S coordinates) equation of the caustic may be found by writing ξ, v for x, y in the equation of this inverse.

Now if, as in Quest. 5131, 5232, it be required that the caustic be a conic, an algebraic relation of the second degree must exist between ξ, v; therefore between x, y in the inverse; thus the deflecting surface must be the inverse of a conic. It is therefore, in general, a bicircular quartic; this is the case of Quest. 5232. If the caustic be

$$(a\;b\;c\;f\;g\;h \between \xi, v, 1)^2 = 0,$$

the surface is

$$c(x^2+y^2)^2 + 2\operatorname{cosec}\alpha\,(x^2+y^2)(fy+gx) + (ax^2+by^2+2hxy)\operatorname{cosec}\alpha = 0,$$

in general a bicircular quartic. But if $a=b$ and $h=0$, it breaks up into $x^2+y^2=0$ and the circle

$$c(x^2+y^2) + 2\operatorname{cosec}\alpha\,(fy+gx) + a\operatorname{cosec}^2\alpha = 0 \quad\text{..................(2),}$$

which is the case of Question 5131.

Putting $f = 0$, $\frac{g}{c} \operatorname{cosec} \alpha = -\lambda$, $\frac{a}{c} \operatorname{cosec}^2 \alpha = \lambda^2 - 1$,

the circle (2) reduces to the form used in Professor TANNER's solution, and the caustic becomes

$$(\lambda^2 - 1) \sin^2 \alpha \, (\xi^2 + \nu^2) - 2\lambda\xi \sin \alpha + 1 = 0,$$

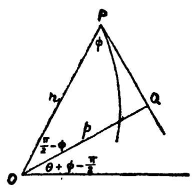
which may be transformed in the usual way into Cartesian coordinates, giving the same result as in the solution alluded to.

4775. (By Professor MINCHIN, M.A.)—If at any point V be the potential of any portion of a curve, and V′ the potential of the corresponding portion of the pedal with regard to the point; prove (1) that

$$V = V' + \log \cot \tfrac{1}{2}\phi_1 \cot \tfrac{1}{2}\phi_2,$$

where ϕ_1 and ϕ_2 are the angles made by the terminal tangents to the given curve with the terminal radii vectores drawn from the given point, and the law of force being that of nature; and (2) that, if applied to the case of an ellipse and its focal pedal, this theorem gives Legendre's transformation of an elliptic function of the first kind.

Solution by J. HAMMOND, B.A.



1. We have

$$p\,(d\theta + d\phi) = r \sin \phi \,(d\theta + d\phi)$$
$$= \sin \phi \,(ds \,.\, \sin \phi + r\, d\phi),$$
$$dp = dr \,.\, \sin \phi + r \cos \phi \, d\phi$$
$$= \cos \phi \,(ds \,.\, \sin \phi + r\, d\phi).$$

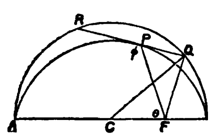
Hence, if ds' be the element of the pedal corresponding to ds, we have

$$ds' = ds \,.\, \sin \phi + r\, d\phi, \quad \text{or} \quad \frac{ds}{r} = \frac{ds'}{p} - \frac{d\phi}{\sin \phi} \;\ldots(1);$$

whence, integrating between limits ϕ_1 and ϕ_2, we obtain

$$V = V' + \log \cot \tfrac{1}{2}\phi_2 - \log \cot \tfrac{1}{2}\phi_1.$$

2. In the case of the ellipse and its focal pedal, the coordinates of P referred to the centre as origin being $a \sin \psi$, $b \cos \psi$,



$$ds = a\,(1 - e^2 \sin^2 \psi)^{\frac{1}{2}}\, d\psi,$$

and

$$r = a\,(1 - e \sin \psi).$$

Thus,

$$\frac{ds}{r} = \frac{1 + e \sin \psi}{(1 - e^2 \sin^2 \psi)^{\frac{1}{2}}}\, d\psi.$$

Now θ being measured from A, $\phi = \mathrm{FPR}$,

p = perpendicular from F $(ae, 0)$ on $\frac{x}{a} \cos \psi + \frac{y}{b} \sin \psi = 1$;

thus, $r \sin \phi = \dfrac{1 - e \sin \psi}{\left(\dfrac{\sin^2 \psi}{a^2} + \dfrac{\cos^2 \psi}{b^2}\right)^{\frac{1}{2}}}$, and $\sin \phi = \left(\dfrac{1 - e^2}{1 - e^2 \sin^2 \psi}\right)^{\frac{1}{2}}$,

and $\cos\phi = \dfrac{-e\cos\psi}{(1-e^2\sin^2\psi)^{\frac{1}{2}}}$; whence $\dfrac{d\phi}{\sin\phi} = -\dfrac{e\sin\psi\,d\psi}{(1-e^2\sin^2\psi)^{\frac{1}{2}}}$;

therefore $$\frac{ds}{r}+\frac{d\phi}{\sin\phi} = \frac{d\psi}{(1-e^2\sin^2\psi)^{\frac{1}{2}}} \qquad\text{.........................(2).}$$

Also, if $ACQ = 2\chi$, we shall have $ds' = 2ad\chi$,

and $p = (a^2+a^2e^2+2a^2e\cos 2\chi)^{\frac{1}{2}} = a(1+e)\left(1-\dfrac{4e}{(1+e)^2}\sin^2\chi\right)^{\frac{1}{2}}$;

therefore $$\frac{ds'}{p} = \frac{2}{1+e}\cdot\frac{d\chi}{\left(1-\dfrac{4e}{(1+e)^2}\sin^2\chi\right)^{\frac{1}{2}}};$$

and from equations (1) and (2) we obtain

$$\frac{d\chi}{(1-k^2\sin^2\chi)^{\frac{1}{2}}} = \frac{1+e}{2}\cdot\frac{d\psi}{(1-e^2\sin^2\psi)^{\frac{1}{2}}};$$

where k is put for $\dfrac{4e}{(1+e)^2}$, so that, if $k'^2 = 1-k^2$, $k' = \dfrac{1-e}{1+e}$, or $e = \dfrac{1-k'}{1+k'}$.

5186. (By Dr. Booth, F.R.S.)—Prove that the locus of the foci of all the elliptic sections of a cylinder, whose planes pass through a tangent to the cylinder parallel to its circular base, is the logocyclic curve.

I. *Solution by* Samuel Roberts, M.A.

The foci lie, of course, in the plane passing through the point of contact of the tangent, and perpendicular to it. If r is the radius of the cylinder, θ the inclination of the cutting plane to the base, the major axis of the section is $2r\sec\theta$, and the minor axis $2r$. Hence the excentricity is $\sin\theta$, and we have immediately

$$\rho = r\sec\theta\,(1\pm\sin\theta),$$

which is the equation of the logocyclic curve, a nodal circular cubic having its node at the centre of the circular cubic through the tangent.

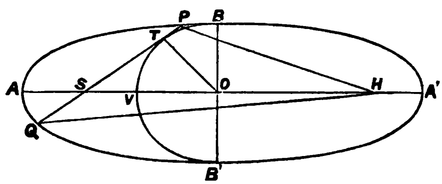
II. *Solution by* J. Hammond, B.A.; C. Leudesdorf, M.A.; *and others.*

Let θ be the inclination of the elliptic section to the circular section, r the distance of either focus measured along the major axis from the generating line, and c the radius of the circular section; then the axes of the elliptic section are $c\sec\theta$ and c, and the distance of the foci from the centre is $c\tan\theta$. Thus the equation of the locus in polar coordinates is $r = c(\sec\theta\pm\tan\theta)$, which, when transformed to rectangular coordinates, is $(x-c)^2(x^2+y^2) = c^2y^2$; or, reducing and dividing by a factor x, it becomes $(x^2+y^2)(x-2c)+c^2x = 0$, the equation to the logocyclic curve.

3656. (By the EDITOR.)—Through one focus S of an ellipse,—whose semiaxes and eccentricity are a, b, e,—a chord PSQ is drawn, and its ends are joined to the other focus H; show that (1) the average area of the triangle HPQ is $\frac{2b^2}{\pi}\log\left(\frac{1+e}{1-e}\right)$; (2) when $e^2 > \frac{1}{2}$, that is, when the foci are outside the circle on the minor axis, the triangle HPQ will be a *maximum* ($= ab$) when its base PQ touches the circle on the minor axis, and a *minimum* ($= 2eb^2$) when its base is perpendicular to the major axis; and (3), when $e^2 =$ or $< \frac{1}{2}$, the triangle HPQ is a *maximum* when its base PR is perpendicular to the major axis.

Solution by the PROPOSER.

Let AOA′, BOB′ be the axes of the ellipse; then, putting θ for the



variable angle PSH, we have

$$\text{Length of chord PQ} = \frac{2b^2}{a(1-e^2\cos^2\theta)},$$

$$\text{Length of perpendicular from H on PQ} = 2ae\sin\theta;$$

$$\text{Area } (\Sigma) \text{ of triangle HPQ} = \frac{2b^2 e\sin\theta}{1-e^2\cos^2\theta}.$$

The average area (Σ_1) of Σ is, therefore,

$$\Sigma_1 = \frac{1}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\Sigma d\theta = \frac{4b^2e}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\theta\, d\theta}{1-e^2\cos^2\theta} = \frac{4b^2}{\pi}\int_0^1\frac{e dx}{1-e^2x^2}$$

$$= \frac{2b^2}{\pi}\left[\log\left(\frac{1+ex}{1-ex}\right)\right]_{x=0}^{x=1} = \frac{2b^2}{\pi}\log\left(\frac{1+e}{1-e}\right).$$

To find the greatest triangle of the series, we have

$$\frac{d\Sigma}{d\theta} = \frac{2b^2e\cos\theta\,(1-e^2-e^2\sin^2\theta)}{(1-e^2\cos^2\theta)^2} = \frac{2b^2e\cos\theta\,(b^2-a^2e^2\sin^2\theta)}{a^2(1-e^2\cos^2\theta)^2},$$

which vanishes when $\theta = \frac{1}{2}\pi$, and likewise when $\sin\theta = \frac{b}{ae} = \frac{(1-e^2)^{\frac{1}{2}}}{e}$.

When $\sin\theta = \frac{b}{ae}$, we have $\Sigma = ab$, which, when possible, is a *maximum* value. But, taking the value of θ from the form $\sin^2\theta = \frac{1-e^2}{e^2}$, it will be seen that this maximum is possible only when $e^2 > \frac{1}{2}$.

The construction for this maximum value is obtained by drawing PSQ to touch at T a semicircle (BVB′) on BB′ as diameter; for then

$$\sin \text{PSH} = \frac{\text{OT}}{\text{OS}} = \frac{b}{ae}.$$

When $\theta = \frac{1}{2}\pi$, we have $\Sigma = 2b^2e$, which, when $e^2 > \frac{1}{2}$, will be a *minimum* between the two maxima (each $= ab$) above and below AA′; but when $e^2 < \frac{1}{2}$, this value will be the only maximum the triangle admits of.

On Temperament in Music.

By R. H. M. Bosanquet, *Fellow of St. John's College, Oxford.*

The labours of Helmholtz have placed the world in possession of a satisfactory theory of consonance and dissonance; *i.e.*, we have now an intelligible conception of the reason why we can tune an octave, a fifth, a major third, a harmonic seventh, and any chords made up of these intervals, as *smooth combinations*, by adjusting the pitch of the notes until beats disappear.

There can be little doubt that scales, and modern tonality, owe their origin to the tuning of the lyre, harp, lute, guitar, &c., so that their strings form consonant or smooth combinations,—by the minstrels or popular singers of all ages.

The requirements of harmonious music compel us to provide notes, which shall constitute the material of a symmetrical system of consonances with respect to any note of the system; such a system may be called *regular.*

If the symmetry be complete, so that we can work round from any starting point back to the same, the system may be called a *regular cyclic* system.

If the symmetry be incomplete, so that we can work from certain starting points regularly for a certain distance only, the system may be called a *regular noncyclic* system.

All systems in which the octave is divided into n equal intervals form regular cyclic systems.

Regular noncyclic systems are combinations of intervals extended up and down from a starting point, according to a uniform law, to any required extent.

Intervals may be conveniently expressed by the logarithms of vibration ratios to any base.

The intervals of the primary consonances, octave, fifth, and major third, are incommensurable with one another.

We shall express intervals in terms of a logarithmic system such that 12 is the logarithm of 2; whence, the octave is divided into 12 equal parts. The numbers of this system may be obtained by multiplying the common logs. of the vibration ratios by the factor $\frac{12}{\log 2}$.

In this notation, the fifth and major third are represented by the following numbers to five places of decimals:—

Fifth = 7·01955 (Ratio = $\frac{3}{2}$),
Third = 4 − ·13686 (Ratio = $\frac{5}{4}$).

No regular system can be formed, which will contain at the same time both accurate fifths and thirds (for that would involve commensurable relations between incommensurable quantities).

Temperaments are systems formed according to any rule involving approximate commensurable relations between fifths, thirds, and octaves, or between fifths and thirds only.

In cyclic temperaments both thirds and fifths are tempered (altered by small quantities); in regular noncyclic temperaments either fifths or thirds may be left perfect at will, but not both.

Ordinary (equal) temperament.—If the decimals in the above values are omitted, the following simple commensurable relations exist between fifth, third, and octave:—

Octave = 12 units, Fifth = 7 units, Third = 4 units.

These units are called "equal temperament semitones"; each is the $\frac{1}{12}$th part of an octave; and the equal temperament semitone is the interval between any two consecutive notes of the piano as ordinarily tuned.

Helmholtz's Theorem.—8 (7·01955) = 56·15640
= 60 − (4 − ·15640).

8 perfect fifths fall short of five octaves (5 × 12) by an interval differing but little from a major third.

Major third	=	4 − ·13686
Eight-fifths third	=	4 − ·15640
Difference	=	·01954 of an equal temperament semitone.

This difference is called a skhisma.

Note its approximate identity with the decimal of the perfect fifth. (Fifth = 7·01955). This is purely accidental, and is one of the most remarkable coincidences in mathematics. It forms the basis of higher approximations, which we cannot enter into here.

This theorem forms the basis of systems of tuning by perfect or approximately perfect fifths, in which the major thirds are arrived at by a chain of 8 fifths.

The derivation of a third by a chain of 8 fifths is contrary to the fundamental conventions of technical music, which require the third to be derived from a chain of 4 fifths. This is one mode of stating an essential difference between the laws of ordinary harmony and the harmony of perfect consonances, which results in the constant occurrence of passages in ordinary simple harmony which cannot be constructed with perfect concords at all. The difficulty is prior to the question of keyboards altogether. Those who profess to be able to play ordinary music in perfect harmony by means of any particular keyboard, profess, according to this view, to have practical methods of solving problems which really have no solution. The fact is, no doubt, that the phenomena are not realised, and the difficulties are not grasped. It is impossible to enter into details on this point without employing musical symbols.

Mean tone system.—In accordance with the conventions of technical music, thirds require to be formed by a chain of 4 fifths. The most

ancient and simple mode of satisfying this requirement * is to temper the four fifths equally, so as to make the third perfect.

Now four fifths are	$4 \times 7{\cdot}01955$	=	$28{\cdot}07820$
	Two octaves and a third	=	$28 - {\cdot}13686$
	Difference	=	${\cdot}21506$

which is the interval called a comma, and may be obtained as the equivalent of the ratio $\frac{81}{80}$. To make the four fifths = 2 octaves + third, each of the four fifths must therefore be diminished by $\frac{1}{4}$ of a comma. This is the rule of the mean tone system. It is so called because its tone is $\frac{1}{2}$ a third, or the arithmetic mean between the major and minor tones of the old diatonic scale, which together make a third.

Divisions of the octave.—We have not space to enter fully into this subject, but the following are fundamental principles of a simple character.

The number of units of any division of the octave by which 12 fifths exceed 7 octaves, is called the order of the system.

The system of 53 is of order $+1$,

$$(12 \times 31 = 372, \qquad 7 \times 53 = 371).$$

The system of 31 is of order -1,

$$(12 \times 18 = 216, \qquad 7 \times 31 = 217).$$

In any system of n divisions in the octave and order r, the departure of the fifth from equal temperament is $\frac{r}{n}$ semitones.

Let δ be the departure of the fifth, so that the fifth $= 7 + \delta$; then, since 12 fifths exceed 7 octaves by r units $\left(\text{unit} = \frac{12}{n}\right)$, we have

$$12\,(7+\delta) - 7 \times 12 = r \cdot \frac{12}{n}, \quad \text{and} \quad \delta = \frac{r}{n}.$$

The fifth of the system of 53 is thus $7\frac{1}{53}$; and the fifth of the system of 31 is $7 - \frac{1}{31}$.

It is easy to verify that the system of 53 differs only by small quantities from a system of perfect fifths, forming a third by a chain of 8 fifths, and that the system of 31 differs only by small quantities from the mean tone system, forming a nearly perfect third by a chain of four fifths.

Enough has been said to shew the nature of the reasoning employed, and to indicate the existence of a large class of regular systems, cyclical and noncyclical, which cannot generally be treated practically without more keys in the octave than the ordinary keyboards afford.

The generalised keyboard devised by the writer, by means of which a large class of regular systems may be controlled, can be described in mathematical language. We shall suppose the keys projected on a horizontal plane, so as to get rid of the complication of the difference of level.

The number of equal temperament semitones is set off as an abscissa; the departure or deviation from equal temperament as an ordinate.

Consider the proposition, 12 fifths make nearly 7 octaves:

$$12 \times 7{\cdot}01955 = 84{\cdot}23460$$

* There can indeed be little doubt that it was from the properties of this system, which was universal in the early days of modern music, that this convention, and indeed our whole musical nomenclature, took its rise.

(·23460 is called the comma of Pythagoras). After going round a chain of 12 fifths, we want a note in a similar position in each octave to the note from which we started, *quâ* abscissa, but yet different. We give the note such an ordinate as to let it be out of the way of the first. One-twelfth of this ordinate is the ordinate difference for every fifth of the series.

This does not tie us to the interval value of the fifths; we may make the fifths anything we please. The generalised keyboard simply frees us from the restriction of the ordinary keyboard, which requires that 12 fifths should be equal to 7 octaves.

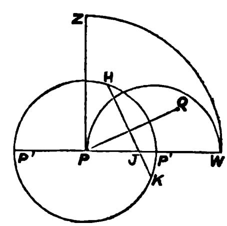
The relative positions of two keys depends only on the interval between them. Hence, any given interval combination has always exactly the same form to the fingers in the same system, no matter in what key or key relationship it is used.

It follows from this symmetrical property that, when the dimensions have been once properly adjusted with respect to the hand, the keyboard is capable of grappling completely with all problems of harmony, of the solution of which any system placed upon it is capable. It differs in this respect from other proposed keyboards founded on key relationship, which are only capable of dealing at all with problems contemplated in the key relationship scheme on which they are founded.

The fingering of the mean tone and allied systems on the generalised keyboard is of very remarkable facility. Passages of the most difficult and complex character can be executed with ease in this manner. The perfect concord systems have another class of fingering on the same keys, which does not so specially lend itself to execution, while it is amply sufficient for dealing with problems in harmony,—the proper function of this keyboard.

5251. (By Colonel A. R. Clarke, C.B., F.R.S.)—Two equal spheres, on each of which a great circle is marked, intersect; show that the chance that the circles are linked is $\frac{1}{4}$.

Solution by the Proposer.



It is easily seen that if the number of great circles on a sphere be 2π, then the number of them which pass between two fixed points on the surface, whose angular distance apart $= \phi$, will be 2ϕ.

Let the spheres have radius equal unity, P being a fixed sphere with its circle P′P′ fixed, and Q being a moveable sphere with its circle in any position. To intersect at all, the centre of Q must be within a sphere concentric with P and having a radius $= 2$. We may confine our attention to one quadrant, ZPP′W, and to that part of the sphere which lies between two consecutive planes passing through ZP and containing an angle $= d\psi$. Now if the centre of the moveable sphere be at Q (in the space just defined), HK bisecting PQ perpendicularly indicates the circle of intersection of the spheres P, Q. If Q be within a semicircle de-

scribed with centre P′ on PP′W, HK will cut the fixed circle PP′ in two points J, J (on opposite sides of the plane of the diagram); but if Q be without this semicircle, the points J, J will be imaginary, and there can be no *linking*. Let $PQ = x$, $QPP' = \theta$; then, if the angular distance apart of the points J, J be ϕ, we have

$$\cos \tfrac{1}{2}\phi = \tfrac{1}{2}x \sec\theta, \quad \text{therefore} \quad 2\phi = 4\cos^{-1}(\tfrac{1}{2}x \sec\theta);$$

and this is the number of circles which link for a single position of Q. The element of volume at Q is $x \cos\theta\, d\psi\, x\, d\theta\, dx$; hence the total number of *linked* circles is

$$\sigma = 4d\psi \int_0^{\frac{1}{2}\pi} \int_0^{2\cos\theta} \cos\theta\, x^2 \cos^{-1}(\tfrac{1}{2}x \sec\theta)\, d\theta\, dx;$$

whilst the total number of circles is (Q ranging over ZPW)

$$\Sigma = 2\pi d\psi \int_0^{\frac{1}{2}\pi} \int_0^2 \cos\theta\, x^2\, d\theta\, dx.$$

σ being transformed to

$$\sigma = \tfrac{32}{3} d\psi \int_0^{\frac{1}{2}\pi} \cos^4\theta\, d\theta \int_0^{\frac{1}{2}\pi} \phi \,.\, 3\cos^2\phi \sin\phi\, d\phi,$$

we readily obtain for the required probability $\dfrac{\sigma}{\Sigma} = \tfrac{1}{4}$.

5254. (By Christine Ladd.)—Find the scalar and the vector of the product of three non-planar Quaternions, and give the resulting formulæ of Plane Trigonometry.

Solution by the Proposer.

Since $\quad Vq'q = Vq'Sq + Sq'Vq + V(Vq'Vq), \quad Sq'q = Sq'Sq + S(Vq'Vq),$

we have

$$\begin{aligned} V(q''q'q) &= Vq'q\,Sq'' + Sq'q\,Vq'' + V(Vq''Vq'q) \\ &= Vq'Sq\,Sq'' + Vq\,Sq'Sq'' + Sq''V(Vq'Vq) \\ &\quad + Vq''Sq'Sq + Vq''S(Vq'Vq) \\ &\quad + V(Vq''Vq'Sq + Vq''Vq\,Sq' + Vq''V\,.\,Vq'Vq), \end{aligned}$$

or, since $V\Sigma = \Sigma V$ and $V\,.\,\gamma V\beta\alpha = \alpha S\beta\gamma - \beta S\gamma\alpha$,

$$\begin{aligned} V\,.\,q''q'q = {} & Vq\,Sq'Sq'' + Vq'Sq''Sq + Vq''Sq'Sq \\ & + Sq V\,.\,Vq''Vq' - Sq'V\,.\,VqVq'' + Sq''V\,.\,Vq'Vq \\ & + VqS\,.\,Vq''Vq' - Vq'S\,.\,VqVq'' + Vq''S\,.\,Vq'Vq \;\ldots\ldots\; (1). \end{aligned}$$

Making the quaternions complanar, substituting their versors, taking the tensor of the resulting equation, and observing that $Vq'' \parallel Vq' \parallel Vq$,

we have $\mathrm{TVU}q''q'q = \mathrm{TVU}q''\mathrm{SU}q'\mathrm{SU}q + \mathrm{SU}q'\,\mathrm{TVU}q'\mathrm{SU}q$
$$+\mathrm{SU}q''\mathrm{SU}q'\mathrm{TVU}q - \mathrm{TVU}q''\mathrm{TVU}q'\mathrm{TVU}q,$$
$$\mathrm{S}(q''q'q) = \mathrm{S}q''\mathrm{S}q'q + \mathrm{S}(\mathrm{V}q''\mathrm{V}q'q) = \mathrm{S}q''\mathrm{S}q'\mathrm{S}q + \mathrm{S}q''\mathrm{S}\,.\,\mathrm{V}q'\mathrm{V}q$$
$$+\mathrm{S}[\mathrm{V}q''\mathrm{V}q'\mathrm{S}q + \mathrm{V}q''\mathrm{V}q\mathrm{S}q' + \mathrm{V}q''\mathrm{V}(\mathrm{V}q'\mathrm{V}q)],$$
or, since $\mathrm{S}\Sigma = \Sigma\mathrm{S}$ and $\mathrm{S}\,.\,\gamma\mathrm{V}\beta\alpha = \gamma\beta\alpha - \gamma\delta\beta\alpha + \beta\delta\gamma\alpha - \alpha\delta\beta\gamma$,
$$\begin{aligned}\mathrm{S}q''q'q &= \mathrm{S}q''\mathrm{S}q'\mathrm{S}q + \mathrm{V}q''\mathrm{V}q'\mathrm{V}q \\ &\quad + \mathrm{S}q''\mathrm{S}(\mathrm{V}q'\mathrm{V}q) - \mathrm{V}q''\mathrm{S}(\mathrm{V}q'\mathrm{V}q) + \mathrm{S}q'\mathrm{S}(\mathrm{V}q\mathrm{V}q'') \\ &\quad + \mathrm{V}q'\mathrm{S}(\mathrm{V}q\mathrm{V}q'') + \mathrm{S}q\mathrm{S}(\mathrm{V}q''\mathrm{V}q') - \mathrm{V}q\mathrm{S}(\mathrm{V}q''\mathrm{V}q') \\ &= \mathrm{S}q''\mathrm{S}q'\mathrm{S}q + \mathrm{V}q''\mathrm{V}q'\mathrm{V}q \\ &\quad + \mathrm{K}q''\mathrm{S}(\mathrm{V}q'\mathrm{V}q) + q'\mathrm{S}(\mathrm{V}q\mathrm{V}q'') + \mathrm{K}q\mathrm{S}(\mathrm{V}q''\mathrm{V}q') \;\;\ldots\ldots\;\; (2).\end{aligned}$$

When the quaternions are complanar and their tensors equal, we have $\mathrm{S}[\mathrm{V}q''\mathrm{V}(\mathrm{V}q'\mathrm{V}q)] = 0$, and $\mathrm{S}(\mathrm{V}q'\mathrm{V}q) = -\mathrm{TV}q'\mathrm{TV}q'$, &c.; hence
$$\mathrm{SU}q''q'q = \mathrm{SU}q''\mathrm{SU}q'\mathrm{SU}q - \mathrm{TVU}q''\mathrm{TVU}q'\mathrm{SU}q$$
$$-\mathrm{TVU}q''\mathrm{SU}q'\mathrm{TVU}q - \mathrm{SU}q''\mathrm{TVU}q'\mathrm{TVU}q.$$

Hence, for any three angles we have
$$\sin(x+y+z) = \sin x\cos y\cos z + \cos x\sin y\cos z$$
$$+\cos x\cos y\sin z - \sin x\sin y\sin z,$$
$$\cos(x+y+z) = \cos x\cos y\cos z - \sin x\sin y\cos z$$
$$-\sin x\cos y\sin z - \cos x\sin y\sin z.$$
These trigonometrical formulæ might, of course, have been obtained more simply, but the equations in non-complanar quaternions are not without value. Adding (1) and (2), we obtain
$$q''q'q = \Pi\mathrm{S} + \Pi\mathrm{V} + \Sigma(\mathrm{VSS}) + \Sigma(\mathrm{SVV}),$$
which serves to confirm the correctness of those equations.

5245. (By Professor Townsend, F.R.S.)—A cone of revolution being supposed to stand on an ellipse of semiaxes a and b, show that the difference of the areas, into which its surface is divided by the plane connecting its vertex with the minor axis of the ellipse, is constant, and $= 2b(a^2-b^2)^{\frac{1}{2}}$ [$= 2abe$, if e be the eccentricity], whatever be the position of its vertex on the focal hyperbola of the ellipse.

I. *Solution by* Dr. Booth, F.R.S.

In the second Volume of Booth's *New Geometrical Methods*, Art. 262, it is shown that the side of a right cone between the vertex and a point on the conic which forms the base of the cone, consists of two parts, a constant part and a variable part. The constant part is that portion of the

side of the cone between its vertex and the point in which it meets the *circle of contact* of the inscribed *focal sphere*. Let us call this constant part t. The variable part of the side of the cone is the distance between the point in which the side meets the *circle of contact* and the given point on the conic which forms the base of the cone. This part may be denoted by ρ. Hence the whole side of the cone is $(t+\rho)$.

In the same Article it is shown that this portion ρ is equal to the focal vector drawn from the point assumed on the elliptic base of the cone to the point of contact of the *focal sphere* with the plane of the elliptic base, or, in other words, its focus. Let a, b, and e be the semiaxes and eccentricity of the elliptic base, and let χ be the angle which the side of the cone makes with the tangent to the elliptic base at the given point. But in Art. 267 of the same volume it is shown that this angle χ is also the angle between the tangent to the conic and the focal vector ρ of this point.

Let $d\Sigma$ be the element of the surface of the cone between two successive sides thereof, indefinitely near to each other; then it is obvious that

$$2d\Sigma = (t+\rho)\tan\chi \, . \, d\rho \quad \text{.........................} \quad (a).$$

Now $\quad \sin^2\chi = \dfrac{b^2}{(2a-\rho)\,\rho}$; $\quad$ hence $\quad \tan\chi = \dfrac{b}{(2a\rho-\rho^2-b^2)^{\frac{1}{2}}}$ ....(b, c);

therefore
$$2\Sigma = \int \frac{b\,(t+\rho)\,d\rho}{(2a\rho-\rho^2-b^2)^{\frac{1}{2}}} \quad \text{.............................} \quad (d).$$

Assume
$$\cos\phi = \frac{a-\rho}{ae} \quad \text{.................................} \quad (e);$$

then the foregoing expression becomes

$$2\Sigma = b\int(t+a)\,d\phi - bae\int\cos\phi\,d\phi \quad \text{.....................} \quad (f),$$

or
$$2\Sigma = b\,(t+a)\,\phi - abe\sin\phi + C \quad \text{........... ....} \quad (g).$$

Let us start from the longest side of the cone. Then at this point $\Sigma = 0$, $\cos\phi = 1$, $\sin\phi = 0$, $\phi = 0$. Therefore the constant $C = 0$.

Let S denote the sheet of the cone between the longest side and the side that passes through the extremity of the minor axis, and and let S_1 be the sheet between this line and the shortest side of the cone. At the end of the side of the cone which ends on the minor axis, we have $\cos\phi = 0$, $\phi = \frac{1}{2}\pi$, $\sin\phi = 1$; therefore

$$2S = \tfrac{1}{2}(t+a)\,b\pi - abe \quad \text{..............................} \quad (h).$$

If we extend the integration from the longest to the shortest side of the cone, we shall have $\cos\phi = -1$, $\sin\phi = 0$, $\phi = \pi$; therefore

$$2\Sigma = 2S + 2S_1 = (t+a)\,b\pi \quad \text{..........................} \quad (i).$$

Subtracting 4S from this equation, we have

$$2S_1 - 2S = 2abe \quad \text{.................................} \quad (j),$$

a value independent of the axis of the cone.

In Art. 281, Cor. i., of the same work it is shown that the volumes of all such cones are equal to $\frac{1}{3}\pi b^3\cot\theta$, or as the cubes of the minor axes of their elliptic bases; θ being the semiangle of the cone.

Eliminating θ, i between the equations

$$ap\tan\theta = b^2, \quad \cos i = e\cos\theta, \quad x\tan i = p\tan\theta,$$

we obtain for the vertex of the cone the hyperbola

$$\frac{x^2}{a^2e^2} - \frac{p^2}{b^2} = 1.$$

II. *Solution by the* PROPOSER.

Denoting by θ the semi-angle of the cone, by r_1 and r_2 the distances of its vertex from the foci of the hyperbola, by p_1, p_2, and p the distances of its axis from the same and from the common centre of the ellipse and hyperbola, and by α the angle made by p_1, p_2 and p with the common transverse axis of both conics; then since, evidently, the difference in question $= 2bp \operatorname{cosec}\theta = 2bp\left(\frac{r_1r_2}{p_1p_2}\right)^{\frac{1}{2}}$, and since, from the hyperbola, $p\,(r_1r_1)^{\frac{1}{2}} = b\,(a^2-b^2)^{\frac{1}{2}}$, and $p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} = b$, therefore, &c.

III. *Solution by* SAMUEL ROBERTS, M.A.

In the first instance, suppose that an oblique cone stands on an elliptic base, and that the foot of the perpendicular to the base from the vertex falls upon the major axis of the ellipse. The axes of the ellipse and a line perpendicular to its plane through the centre being taken as axes of coordinates, let the coordinates of the vertex be (O, X, Z), and those of P a point on the ellipse ($b\cos\phi$, $a\sin\phi$, O), then, if D is the perpendicular from the vertex of the cone to the tangent at P, we have

$$\frac{dS}{d\phi} = \tfrac{1}{2}D\frac{ds}{d\phi},$$

the surface S and the arc s being measured respectively from a line through the vertex and an extremity of the minor axis, and from that extremity. In terms of the excentric angle ϕ, the length of the perpendicular on the tangent at P from the foot of the perpendicular on the base from the vertex is $\dfrac{b\,(a-X\sin\phi)}{\{a^2-(a^2-b^2)\sin^2\phi\}^{\frac{1}{2}}}$ and $\dfrac{ds}{d\phi} = \{a^2-(a^2-b^2)\}^{\frac{1}{2}}\sin^2\phi$. We have

therefore $$\frac{dS}{d\phi} = \tfrac{1}{2}\left\{Z^2\left[a^2-(a^2-b^2)\sin^2\phi\right]+b^2\,(a-X\sin\phi)^2\right\}^{\frac{1}{2}}.$$

The integral is generally elliptic, but if the condition for a right cone, viz. $\dfrac{X^2}{a^2-b^2}-\dfrac{Z^2}{b^2} = 1$, is satisfied, the quantity under the radical sign is a perfect square, and

$$\frac{dS}{d\phi} = \tfrac{1}{2}\left\{\frac{abX}{(a^2-b^2)^{\frac{1}{2}}} - b\,(a^2-b^2)\,\tfrac{1}{2}\sin\phi\right\}.$$

Integrating from 0 to π and from π to 2π, we get the relation required, which is independent of X.

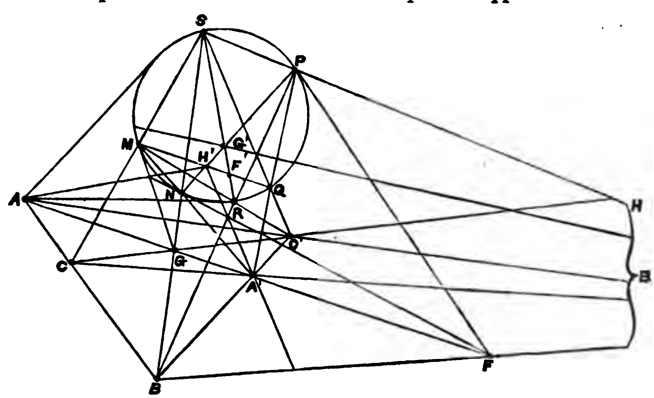
We can at once see that, if the difference in question is constant, it is equal to twice the triangle formed by a focus and the extremities of the minor axis. For the focal hyperbola passes through the foci of the ellipse,

and therefore the flat surface generated by focal vectors is a limiting case of a right cone enveloping the ellipse. It is therefore possible to avoid integration by shewing that the difference of the perpendiculars from the vertex of a right cone to the tangents at the extremities of a diameter of the ellipse, is independent of the position of the vertex on the focal hyperbola (this follows from the corresponding values of D).

5193. (By T. Cotterill, M.A.)—A conic and a quadrilateral are conjugate; show that the polar triangle of the diagonals of the quadrilateral cuts the conic in six points, to which the sides of the quadrilateral are Pascal lines; and explain the connection of the theorem with the following problem:—"Through three points to draw the sides of two triangles inscribed in a conic."

Solution by Professor F. Armenante.

The two points of intersection of the two pairs of opposite sides of the



quadrangle MNPQ, inscribed in a conic, are situated in FG, the polar of H′ with respect to the conic. But the points A, A′ are conjugate and are situated on FG, the polar of H′; hence the triangle A′AH′ is self-conjugate with respect to the conic and will be the diagonal triangle of the quadrangle MNPQ: hence the opposite sides of the quadrangle MNPQ pass through the points A, A′. In like manner the other pairs of opposite sides of the quadrangle MSQR pass through C, C′; and the other two pairs of opposite sides of the quadrangle NPSR pass through B, B′. Therefore the sides of the quadrilateral (AA′, BB′, CC′) are Pascal lines with respect to the exagramm of the conic. Hence we see that if there are given three points A′, B′, C′, and it be required to inscribe in a conic two triangles whose sides shall pass through these three points, it will be

necessary to find three other points A, B, C, conjugates of A′, B′, C′, with respect to the conic, then to construct the diagonal triangle of the quadrilateral (AA′, BB′, CC′), and to find the polar reciprocal triangle of the diagonal triangle. The reciprocal polar triangle will cut the conic in the two triangles which will satisfy the conditions of the problem.

5230. (By C. W. MERRIFIELD, F.R.S.)—If U be a quartic, H its Hessian, and c_1, c_2, c_3 the roots of the cubic $4c^3 - Sc + T = 0$; prove that $\Sigma (c_2 - c_3)(c_1 H - c_1{}^2 U)^{\frac{1}{2}}$ is a perfect square, and its root is a root of the Hessian.

I. *Solution by the* PROPOSER.

The work is exactly the same as in SALMON's *Higher Algebra*, 3rd edition, pp. 192, 193, Art. 212, only that here we have

$$\alpha^2 = c(1-9c^2),$$
$$\beta^2 = -\tfrac{1}{16}(c+1)(3c-1)^2(3c+1),$$
$$\gamma^2 = -\tfrac{1}{16}(c-1)(3c+1)^2(3c-1),$$
$$\beta^2 - \gamma^2 = \frac{1-9c^2}{16}\Big\{(c+1)(3c-1)-(c-1)(3c+1)\Big\} = \tfrac{1}{4}c(1-9c^2) = \tfrac{1}{4}\alpha^2.$$

Evidently, if H vanishes, the sum of the quantities under Σ vanishes identically.

II. *Solution by* SAMUEL ROBERTS, M.A.

The expression vanishes if $H = 0$, shewing that it contains a factor of H.

Using the canonical form $x^4 + 6cx^2y^2 + y^4$, the equation $4c^3 - Sc + T = 0$ becomes $4Z^3 - 2(1+3c^2) + c - c^3 = 0$, the roots of which are

$$c, \quad -\tfrac{1}{2}(c+1), \quad -\tfrac{1}{2}(c-1).$$

The values of $cH - c^2U$ are respectively

$$c(1-9c^2)x^2y^2, \quad -\tfrac{1}{2}(c+1)\tfrac{1}{2}(3c+1)(x^2+y^2)^2,$$
$$-\tfrac{1}{2}(c-1)\tfrac{1}{2}(3c-1)(x^2-y^2)^2.$$

Now $\alpha xy + \beta(x^2+y^2) + \gamma(x^2-y^2)$ is a perfect square of $\alpha^2 = 4(\beta^2-\gamma^2)$.

We have

$$(1-9c^2)c = -\tfrac{1}{4}\left\{\begin{matrix}(3c+1)(3c-1)^2(c+1) \\ -(3c-1)(3c+1)^2(c-1)\end{matrix}\right\}$$
$$= -\tfrac{1}{4}\left\{\begin{matrix}(9c^2-1)(3c^2+2c-1) \\ -(9c^2-1)(3c^2-2c-1)\end{matrix}\right\},$$

and the required condition is satisfied.

5246. (By Professor CROFTON, F.R.S.)—A man stands at the centre of a long plank which is floating on water; find how far he may walk towards either end without upsetting the plank (the thickness being supposed very small).

I. *Solution by* Colonel A. R. CLARKE, C.B., F.R.S.

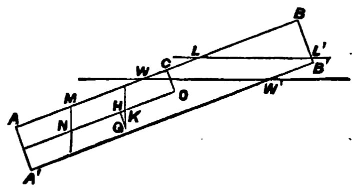
Let a, b, c be the length, breadth, and thickness of the plank; W, σ its weight and specific gravity, w the weight of the man, the unit of weight being the unit cube of water. If h be the depth to which the plank is submerged when the man stands at its centre, then if $h+k=c$, we have

$$abh = abc\sigma + w,$$

$$abk = abc(1-\sigma)\,w.$$

As k must be positive, we have the necessary condition

$$w < abc(1-\sigma); \text{ or, } a > \frac{w}{bc(1-\sigma)}.$$



The accompanying figure represents a vertical section of the plank through its axis; OH, HG are the coordinates (along and perpendicular to the axis ON) of the centre of gravity G of the submerged portion of the plank, viz., AWW′A′, or LAA′B′L′ as the case may be. Representing the man by a heavy particle M, draw MN, GK vertical, then for equilibrium $\mathrm{OK}(\mathrm{W}+w) = \mathrm{ON}w$. If $\mathrm{CM} = \mathrm{U}$ and θ be the inclination of the plank, this becomes $(\mathrm{OH}-\mathrm{HG}\tan\theta)(\mathrm{W}+w) = (\mathrm{U}+\frac{1}{2}c\tan\theta)\,w$. Two cases arise, (1) when WW′ is the water line; (2) when it is LL′. In the case of WW′ we find the coordinates of the centre of gravity G to be

$$\mathrm{OH} = \frac{ak}{2c} - \frac{c^3\cot^2\theta}{24ah}; \quad \mathrm{GH} = \frac{c^3\cot\theta}{12ah}.$$

Substituting these in the equation given, there results

$$\mathrm{U} = \frac{ak}{2c}\cdot\frac{\mathrm{W}+w}{w} - \frac{bc^3}{12w} - \frac{bc^3}{24w}\cot^2\theta - \tfrac{1}{2}c\tan\theta.$$

The value of θ which makes this a maximum is

$$\tan\theta = \left(\frac{bc^2}{6w}\right)^{\frac{1}{3}},$$

and the maximum value of CM is therefore

$$\mathrm{U} = \frac{ak}{2c}\cdot\frac{\mathrm{W}+w}{w} - \frac{bc^3}{12w} - \tfrac{3}{4}c\left(\frac{bc^2}{6w}\right)^{\frac{1}{3}}.$$

The length W′B′ of the under surface which is out of water is

$$\frac{ak}{c} - \tfrac{1}{2}c\left(\frac{6w}{bc^2}\right)^{\frac{1}{3}}.$$

This being positive, we have $ak > \tfrac{1}{2}c^2\left(\frac{6w}{bc^2}\right)^{\frac{1}{3}}$,

and therefore, for case 1st, the necessary condition is that

$$a(1-\sigma) > \frac{w}{bc} + \tfrac{1}{2}c\left(\frac{6w}{bc^2}\right)^{\frac{1}{3}}.$$

But (case 2) when $a(1-\sigma)$ lies between

$$\frac{w}{bc} \text{ and } \frac{w}{bc}+\tfrac{1}{2}c\left(\frac{6w}{bc^3}\right)^{\frac{1}{2}},$$

the plank will upset when the water line is in the position LL′. In the 2nd case, we have for OH, HG the values

$$\text{OH} = \frac{k}{h}\left(\tfrac{1}{2}a-\tfrac{1}{3}v\right); \quad \text{HG} = \frac{k}{h}\left(\tfrac{1}{2}c-\tfrac{1}{3}u\right).$$

where $u = \text{BL}'$, $v = \text{BL}$. Put $x^2 = \tan\theta$, then $uv = 2ak$, $u = vx^2$. Substituting in the equation of equilibrium

$$(\text{OH}-\text{HG}\,.\,x^2)\,(\text{W}+w) = (u+\tfrac{1}{2}cx^2)\,w,$$

and reducing, we get

$$u = \frac{abk}{w}\left\{\tfrac{1}{2}a+\tfrac{1}{3}(2ak)^{\frac{1}{2}}\text{X}\right\}, \text{ where } \text{X} = -\frac{1}{x}-\frac{3ac^2(1-\sigma)}{(2ak)^{\frac{3}{2}}}x^2+x^3.$$

When this is a maximum, we have

$$\frac{d\text{X}}{dx} = \frac{1}{x^2}-\frac{6ac^2(1-\sigma)}{(2ak)^{\frac{3}{2}}}x+3x^2$$

equal to zero. From this biquadratic we cannot get an expression for θ, but as the higher powers of x are necessarily small quantities, we can get an approximation; putting the equation in the form

$$1-\frac{x^3}{n^3}+3x^4 = 0,$$

we get $x = n+n^5$, whence the maximum value of X is

$$\text{X} = -\tfrac{3}{2}\,\frac{1}{n}+n^3,$$

and the corresponding value of u is easily found to be

$$u = \frac{a^2bk}{2w}\left\{1-\left[6\frac{c^2}{a^2}(1-\sigma)\right]^{\frac{1}{2}}+\frac{4k^2}{9c^2(1-\sigma)}\right\}.$$

If we would ascertain the condition that the plank should upset at the instant that the upper surface first touches the water, we must make $x^2 = 2\dfrac{k}{a}$ in the equation $\dfrac{d\text{X}}{dx} = 0$; this gives

$$1-6\frac{c^2}{a^2}(1-\sigma)+12\frac{k^2}{a^2} = 0,$$

which may be otherwise expressed thus,

$$2^{\frac{1}{2}}\left(1-\sigma-\frac{w}{abc}\right) = \left(1-\sigma-\frac{a^2}{6c^2}\right)^{\frac{1}{2}}.$$

It is clear that this is impossible, and u increases with θ as the upper surface enters the water. This investigation has supposed the plank to be more than half submerged. If it be *less*, then it is easy to shew the plank cannot upset when the upper surface touches the water, as long as

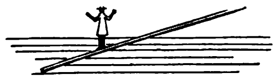
$$a\sigma+\frac{w}{bc} > \tfrac{1}{2}c\left(\frac{6w}{bc^3}\right)^{\frac{1}{2}}.$$

When the plank upsets, at the instant of the under surface commencing to

leave the water, the value of u is determined by the equation

$$2u = \frac{3a}{N^2} - \frac{c}{N^3} - \frac{3c}{N}, \text{ where } \frac{6a}{c}(1-\sigma) = 3N + N^3, \quad N^3 = \frac{6w}{bc^2}.$$

[In the foregoing solution, the man is supposed to stand always *upright*,

thus  however much the plank slopes; or

what comes to the same thing, he is replaced by a heavy particle, thus,

. Under such circumstances, he would certainly *slip* off.

The man may, however, otherwise be supposed to stand always *perpendicular* to the plank, thus  however much the plank slopes; and then, of course, he would *tumble off*; so that either way the man would be in a fix.

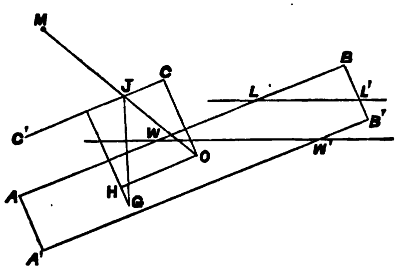
The problem may be more generally enunciated as follows:—

"A rectangular parallelepiped, floating, has a heavy particle attached to a point on one of its faces; determine its position of equilibrium."

[Of the second way of walking the plank, Col. Clarke gives the following solution:—]

2. Let the length AB of plank $= a$; its thickness $AA' = c$; and W, w the weights of the plank and of the man. Let M be the centre of gravity of the man at a given height above axis, OH, of the plank. Supposing the man to retain a position perpendicular to the plank (which simplifies the solution) the locus of the centre of gravity of the man and plank will be a line CC′ parallel to AB and at a height $CO = g$. Let OH, HG be the coordinates of the centre of gravity G of the submerged portion of the plank measured from the centre of the axis. When the man stands at the centre of the plank, let h be the depth to which the plank is submerged, and let $h + k = c$. Let σ = specific gravity of the plank, then we have

$$\frac{h}{c} = \sigma\left(1 + \frac{w}{W}\right).$$



The question divides itself into two cases, first when the water line is in the position WW′, and again when it is in the position LL′ at the moment of upsetting. Let the inclination BWW′ or BLL′ be $= \theta$, then since the area of the unsubmerged (sectional) area is ck, the cases above correspond to, (1) $\theta > \tan^{-1}\frac{c^2}{2ak}$; (2) $\theta < \tan^{-1}\frac{c^2}{2ak}$. Let a vertical line through G meet CC′ in 3, then, if this be also the intersection of OM with CC′, there is equilibrium and the distance moved by the man is

CJ . (W + w) ÷ w. We have then to find the maximum value of

$$CJ = OH-(g+HG)\tan\theta, \text{ or } CJ = \frac{ak}{2c}-g\tan\theta-\frac{c^3}{24ah}\cot^2\theta-\frac{c^3}{12ah}.$$

For the maximum value $\tan^3\theta = \frac{c^3}{12agh}$, and the maximum value of CJ is at once expressed, from which we get the greatest distance to which the man can walk $\frac{W+w}{w}\left\{\frac{ak}{2c}-\frac{c^3}{12ah}-\frac{3}{2}\left(\frac{g^2c^3}{12ah}\right)^{\frac{1}{3}}\right\}$.

The criterion for this case is

$$\tan\theta = \frac{c}{(12agh)^{\frac{1}{3}}} > \frac{c^2}{2ak}, \text{ therefore } 2a^2k^3 > 3ghc^3.$$

But if $3ghc^3 > 2a^2k^3$, we have case (2) where LL′ is the water line. Here the coordinates OH, HG are differently expressed from those in the former case. In fact putting for a moment BL′ = u, BL = v,

$$OH = \frac{k}{h}\left(\frac{a}{2}-\frac{v}{3}\right), \quad HG = \frac{k}{h}\left(\frac{c}{2}-\frac{u}{3}\right);$$

let $\tan\theta = x^2$, then, since $uv = 2ak$, and $u = v\tan\theta$,

$$u = x(2ak)^{\frac{1}{2}}, \quad v = \frac{1}{x}(2ak)^{\frac{1}{2}},$$

and $CJ = OH-(HG+g)\tan\theta$ is

$$= \frac{ak}{2h}-\frac{k(2ak)^{\frac{1}{2}}}{3h}\left\{\frac{1}{x}+\frac{1}{2}\left(\frac{6gh+3kc}{k(2ak)^{\frac{1}{2}}}\right)x^2-x^3\right\}.$$

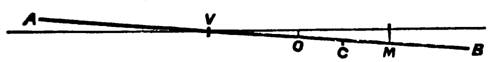
Differentiating with respect to x for a maximum, we get a result of the

form $$-\frac{1}{x^2}+\frac{x}{n^3}-3x^2 = 0.$$

From which we cannot obtain x in finite terms. But as the higher powers of x are small, we may omit them, and get the approximation $x = n + n^5$, whence, and putting $6gh+3kc = G$, we get for the greatest distance that can be passed over by the man

$$\left(\frac{W+w}{w}\right)\frac{k}{h}\left\{\frac{a}{2}-\left(\frac{a}{4}G\right)^{\frac{1}{2}}+\frac{2ak^2}{3G}\right\}.$$

II. *Solution by the* PROPOSER.



Let AB = $2a$ = length of plank; O middle point; σ = specific gravity; W = weight of plank = $2a\sigma$; w = weight of man = $m\sigma$ (so that the man weighs as much as a length m of plank).

Put OM = x; when on the point of upsetting, the man is at M; VB is wholly immersed, and VA out of water.

Let C bisect VB: then we have

$$VB \,.\, 1 = 2a\sigma + m\sigma; \quad \text{therefore} \quad VB = (2a+m)\sigma.$$

Taking moments round V, we have $(W+w)\,VC = W \,.\, VO + w \,.\, VM,$

$$(2a+m)\,\sigma\,(a+\tfrac{1}{2}m)\,\sigma = 2a\sigma\,(2a\sigma+m\sigma-a)+m\sigma\,(2a\sigma+m\sigma-a+x),$$

therefore $$x = \frac{2a+m}{2m}\left[2a-(2a+m)\,\sigma\right].$$

If the man can go to the end of the plank, we have $x = a$;
$\therefore\ 2am = 2a\,(2a+m)-\sigma\,(2a+m)^2,\ 0 = 4a^2-\sigma\,(2a+m)^2,\ \therefore\ 2a = (2a+m)\sigma^{\frac{1}{2}},$
$\therefore\ m = \dfrac{2a}{\sigma^{\frac{1}{2}}}-2a,\ \therefore$ weight of man $\leqq$ weight of plank $\left(\dfrac{1}{\sigma^{\frac{1}{2}}}-1\right)$. For example, if specific gravity $=\frac{1}{4}$, weight of man $=$ weight of plank.

III. *Solution by* G. S. Carr.

As the man advances towards one end of the plank the line of the resultant fluid pressure advances also, coinciding always with the line of the resultant weight until the upper edge of one end of the plank touches the surface. If the man now advances farther the plank will upset, for the former resultant begins to recede from the end, as is easily shown, and therefore can no longer be in equilibrium with the resultant weight.

Let a be the length of the plank, d the thickness, W its weight, w the weight of the man, and c the depth of plank submerged when the man stands over the centre.

When the upper edge of the plank touches the water, let x be the length under water measured along the under surface, and y the distance of the man from the submerged end. Then $\frac{1}{2}dx = ac$, the constant volume submerged.

Taking moments about the submerged end, we have, neglecting the deflection from the horizontal, which is very small,

$$\tfrac{1}{3}(W+w)\,x = \tfrac{1}{2}Wa+wy;$$

therefore $$y = a\left\{\left(\frac{W}{w}+1\right)\frac{2c}{3d}-\frac{W}{2w}\right\}.$$

Now y will be negative if $\dfrac{w}{W}$ be $<\dfrac{3d}{4c}-1$, in which case the man will always be able to walk to the end of the plank without upsetting it.

The ratio $\dfrac{c}{d}$, both of whose terms are very small, is equal to the joint specific gravity of the plank and the man, water being the standard. If S, s be the specific gravities of the plank and the man respectively, then

$$\frac{d}{c} = \frac{Ws+wS}{Ss\,(W+w)}.$$

5125. (By J. J. Walker, M.A.)—If A, B, C be any three points on a straight line, and A′ be taken harmonic to A with respect to B and C, B′ to B with respect to C and A, C′ to C with respect to A and B: prove (1) that A is harmonic to A′ with respect to B′, C′, &c. &c.; (2) that the six points form three other involutions in which AA′, BC′, B′C are corresponding pairs, or BB′, AC′, A′C, or CC′, AB′, A′B.

I. *Solution by* Professor TOWNSEND, F.R.S.

The original triad of points A, B, C being supposed on a circle or on a conic instead of on a straight line, which will of course in no way affect the result, the three tangents at them to the curve will form a triangle PQR in homology with the triangle ABC, and the three connectors AP, BQ, CR of the three pairs of corresponding vertices of the two triangles will intersect the curve at the derived triad of points A′, B′, C′, and each other at the centre of homology O of the triangles; hence, the three pairs of corresponding points A and A′, B and B′, C and C′, being in perspective, and therefore in involution, on the curve, and the three pairs of corresponding tetrads BCAA′ and B′C′A′A, CABB′ and C′A′B′B, ABCC′ and A′B′C′C being consequently equi-anharmonic, therefore, &c.

II. *Solution by* Professor EVANS, M.A.; L. W. JONES; *and others.*

$$\underline{\text{A} \quad \text{C}' \quad \text{B} \quad \text{A}' \quad \text{C} \quad \text{B}'}.$$

1. Denote by b, c, a', b', c' the distances from A of B, C, A′, B′, C′.

By supposition we have

$$\frac{2}{a'} = \frac{1}{b} + \frac{1}{c}, \quad \frac{2}{c} = \frac{1}{b} + \frac{1}{b'}, \quad \frac{2}{b} = \frac{1}{c} + \frac{1}{c'} \quad \text{.........} \quad (1, 2, 3).$$

Therefore, adding and omitting $\frac{2}{b} + \frac{2}{c}$ from each side, we have

$$\frac{2}{a'} = \frac{1}{b'} + \frac{1}{c'},$$

which shows that A′ is the harmonic conjugate of A with regard to B′C′. Thus AA′ are the double points of the involution determined by BC, B′C′.

2. Hence $\{ABA'C\} = \{AC'A'B'\} = \{A'C'AB'\}$, since the ranges are harmonic. Therefore AA′, BC′, B′C are conjugate pairs in involution. Similarly for the others.

Printed by C. F. Hodgson and Son, 1, Gough Square, Fleet Street, E.C.